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Adam Parusinski, Armin Rainer

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OPTIMAL REGULARITY OF ROOTS OF POLYNOMIALS

ADAM PARUSIŃSKI AND ARMIN RAINER

ABSTRACT. We study the regularity of the roots of complex univariate polynomials whose coefficients depend smoothly on parameters. We show that any continuous choice of the roots of a $C^{n-1,1}$ -curve of monic polynomials of degree n is locally absolutely continuous with locally p -integrable derivatives for every $1 \leq p < n/(n-1)$, uniformly with respect to the coefficients. This result is optimal: in general, the derivatives of the roots of a smooth curve of monic polynomials of degree n are not locally $n/(n-1)$ -integrable, and the roots may have locally unbounded variation if the coefficients are only of class $C^{n-1,\alpha}$ for $\alpha < 1$. We also give a generalization of Ghisi and Gobbino's higher order Glaeser inequalities.

1. INTRODUCTION

This paper is dedicated to the problem of determining the optimal regularity of the roots of univariate polynomials whose coefficients depend smoothly on parameters. There is a vast literature on this problem, but most contributions treat special cases:

- the polynomial is assumed to have only real roots ([8], [27], [43], [1], [20], [5], [6], [42], [7], [12], [31]),
- only radicals of functions are considered ([16], [10], [41], [11], [15]),
- it is assumed that the roots meet only of finite order, e.g., if the coefficients are real analytic or in some other quasianalytic class, ([9], [33], [34], [35], [38]),
- quadratic and cubic polynomials ([39]), etc.

In this paper we consider the general case: let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$(1.1) \quad P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t)Z^{n-j}, \quad t \in (\alpha, \beta),$$

be a monic polynomial whose coefficients are complex valued smooth functions $a_j : (\alpha, \beta) \rightarrow \mathbb{C}$, $j = 1, \dots, n$. It is not hard to see that P_a always admits a continuous system of roots (e.g. [19, Ch. II Theorem 5.2]), but in general the roots cannot satisfy a local Lipschitz condition. For a long time it was unclear whether the roots of P_a admit locally absolutely continuous parameterizations. This question was affirmatively solved in our recent paper [29]: we proved that there is a positive integer $k = k(n)$ and a rational number $p = p(n) > 1$ such that, if the coefficients are of class C^k , then each continuous root λ is locally absolutely

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continuous with derivative λ' being locally q -integrable for each $1 \leq q < p$, uniformly with respect to the coefficients. See the introduction of [29] for the history of the problem and for applications. The main tool of [29] was the resolution of singularities. With this technique we could not determine the optimal parameters k and p .

In the present paper we prove the optimal result by elementary methods. Our main result is the following theorem.

Theorem 1. *Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let P_a be a monic polynomial (1.1) with coefficients $a_j \in C^{n-1,1}([\alpha, \beta])$, $j = 1, \dots, n$. Let $\lambda \in C^0((\alpha, \beta))$ be a continuous root of P_a on (α, β) . Then λ is absolutely continuous on (α, β) and belongs to the Sobolev space $W^{1,p}((\alpha, \beta))$ for every $1 \leq p < n/(n-1)$. The derivative λ' satisfies*

$$(1.2) \quad \|\lambda'\|_{L^p((\alpha, \beta))} \leq C(n, p) \max\{1, (\beta - \alpha)^{1/p}, (\beta - \alpha)^{-1+1/p}\} \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}([\alpha, \beta])}^{1/j},$$

where the constant $C(n, p)$ depends only on n and p .

Remark. The factor $(\beta - \alpha)^{-1+1/p}$, which makes the bound (1.2) blow up if the length of the interval (α, β) tends to 0 unless $p = 1$, appears only in very special situations. For details and more precise bounds see Section 8.7.

This result is best possible in the following sense:

- In general the roots of a polynomial of degree n cannot lie locally in $W^{1,n/(n-1)}$, even when the coefficients are real analytic. For instance, $Z^n = t$, $t \in \mathbb{R}$.
- If the coefficients are just in $C^{n-1,\delta}([\alpha, \beta])$ for every $\delta < 1$, then the roots need not have bounded variation in (α, β) . See [15, Example 4.4].

A curve of complex monic polynomials (1.1) admits a continuous choice of its roots. This is no longer true if the dimension of the parameter space is at least two. In that case monodromy may prevent the existence of continuous roots. We get however the following multiparameter result, where we impose the existence of a continuous root.

Theorem 2. *Let $U \subseteq \mathbb{R}^m$ be open and let*

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x) Z^{n-j}, \quad x \in U,$$

be a monic polynomial with coefficients $a_j \in C^{n-1,1}(U)$, $j = 1, \dots, n$. Let $\lambda \in C^0(V)$ be a root of P_a on a relatively compact open subset $V \Subset U$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \leq p < n/(n-1)$. The distributional gradient $\nabla \lambda$ satisfies

$$(1.3) \quad \|\nabla \lambda\|_{L^p(V)} \leq C(m, n, p, \mathcal{K}) \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}(\overline{W})}^{1/j},$$

where \mathcal{K} is any finite cover of \overline{V} by open boxes $\prod_{i=1}^m (\alpha_i, \beta_i)$ contained in U and $W = \bigcup \mathcal{K}$; the constant $C(m, n, p, \mathcal{K})$ depends only on m , n , p , and the cover \mathcal{K} .

The proof of Theorem 1 makes essential use of the recent result of Ghisi and Gobbino [15] who found the optimal regularity of radicals of functions (we will need a version for complex valued functions; see Section 3):

Theorem 3 (Ghisi and Gobbino [15]). *Let k be a positive integer, let $\alpha \in (0, 1]$, let $I \subseteq \mathbb{R}$ be an open bounded interval, and let $f : I \rightarrow \mathbb{R}$ be a function. Assume that f is continuous and that there exists $g \in C^{k,\alpha}(\bar{I}, \mathbb{R})$ such that*

$$|f|^{k+\alpha} = |g|.$$

Let p be defined by

$$\frac{1}{p} + \frac{1}{k+\alpha} = 1.$$

Then we have $f' \in L_w^p(I)$ and

$$(1.4) \quad \|f'\|_{p,w,I} \leq C(k) \max \left\{ \left(\text{Höld}_{\alpha,I}(g^{(k)}) \right)^{1/(k+\alpha)} |I|^{1/p}, \|g'\|_{L^\infty(I)}^{1/(k+\alpha)} \right\},$$

where $C(k)$ is a constant that depends only on k .

Here $L_w^p(I)$ denotes the weak Lebesgue space equipped with the quasinorm $\|\cdot\|_{p,w,I}$ (see Section 2.2), and $\text{Höld}_{\alpha,I}(g^{(k)})$ is the α -Hölder constant of $g^{(k)}$ on I .

Let us briefly describe the strategy of our proof of Theorem 1. It is by induction on the degree of the polynomial and its heart is Proposition 3. First we reduce the polynomial P_a to Tschirnhausen form $P_{\tilde{a}}$ (indicated by adding *tilde*), where $\tilde{a}_1 \equiv 0$ (see Section 4.1), and split it near points t_0 , where not all coefficients vanish,

$$P_{\tilde{a}}(t) = P_b(t)P_{b^*}(t), \quad t \in I, \quad (t_0 \in I).$$

The splitting is universal and gives formulas for the coefficients b_i (and b_i^*) in terms of \tilde{a}_j (see Section 4.2); hereby the differentiability class is preserved. After Tschirnhausen transformation, $P_b \rightsquigarrow P_{\tilde{b}}$, we split $P_{\tilde{b}}$ near points $t_1 \in I$, where not all \tilde{b}_i vanish,

$$P_{\tilde{b}}(t) = P_c(t)P_{c^*}(t), \quad t \in J, \quad (t_1 \in J).$$

Again the splitting is universal, we get formulas for c_h (and c_h^*) in terms of \tilde{b}_j , and the differentiability class is preserved. Apply the Tschirnhausen transformation, $P_c \rightsquigarrow P_{\tilde{c}}$. Let $k \in \{2, \dots, n\}$ be such that

$$|\tilde{a}_k(t_0)|^{1/k} = \max_{2 \leq j \leq n} |\tilde{a}_j(t_0)|^{1/j}$$

and $\ell \in \{2, \dots, \deg P_b\}$ such that

$$|\tilde{b}_\ell(t_1)|^{1/\ell} = \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(t_1)|^{1/i}.$$

The central idea consists in showing that, for $1 \leq p < n/(n-1)$, we have an estimate of the form

$$(1.5) \quad \| |J|^{-1} |\tilde{b}_\ell(t_1)|^{1/\ell} \|_{L^p(J)} + \sum_{h=2}^{\deg P_c} \| (\tilde{c}_h^{1/h})' \|_{L^p(J)} \leq C \left(\| |J|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J)} + \sum_{i=2}^{\deg P_b} \| (\tilde{b}_i^{1/i})' \|_{L^p(J)} \right),$$

for a universal constant $C = C(n, p)$. In the derivation of this estimate we make essential use of (1.4) in order to bound $\| (\tilde{c}_h^{1/h})' \|_{L^p(J)}$. We get the estimate (1.5) on neighborhoods J of all points $t_1 \in I$, where not all \tilde{b}_i vanish. In order to glue them we prove in Proposition 2 that there is a countable subcollection of intervals J such that every point in their union is covered

at most by two intervals. In this gluing process we use the σ -additivity of $\|\cdot\|_{L^p}^p$. Since the L_w^p -quasinorm lacks this property, we are forced to switch from $L_w^{n/(n-1)}$ - to L^p -bounds for $p < n/(n-1)$.

The paper is structured as follows. We fix notation and recall facts on function spaces in Section 2. Ghisi and Gobbino's result on radicals (Theorem 3) is extended to complex valued functions in Section 3. We collect preliminaries on polynomials and define a universal splitting of such in Section 4. We derive bounds for the coefficients of a polynomial and generalize Ghisi and Gobbino's higher order Glaeser inequalities [15, Prop. 3.4] in Section 4.4, by applying these bounds to the Taylor polynomial. In Sections 5 and 6 we deduce estimates for the iterated derivatives of the coefficients before and after the splitting. Section 7 is dedicated to the proof of Proposition 2. The proof of Theorem 1 is finally carried out in Section 8. We deduce Theorem 2 in Section 9. Finally, in Section 10 we provide three applications of our results: local solvability of a system of pseudodifferential equations, lifting mappings from orbit spaces, and multi-valued Sobolev spaces.

2. FUNCTION SPACES

In this section we fix notation for function spaces and recall well-known facts.

2.1. Hölder spaces. Let $\Omega \subseteq \mathbb{R}^n$ be open. We denote by $C^0(\Omega)$ the space of continuous complex valued functions on Ω . For $k \in \mathbb{N} \cup \{\infty\}$ we set

$$\begin{aligned} C^k(\Omega) &= \{f \in C^0(\Omega) : \partial^\alpha f \in C^0(\Omega), 0 \leq |\alpha| \leq k\}, \\ C^k(\overline{\Omega}) &= \{f \in C^k(\Omega) : \partial^\alpha f \text{ has a continuous extension to } \overline{\Omega}, 0 \leq |\alpha| \leq k\}. \end{aligned}$$

For $\alpha \in (0, 1]$ a function $f : \Omega \rightarrow \mathbb{C}$ belongs to $C^{0,\alpha}(\overline{\Omega})$ if it is α -Hölder continuous in Ω , i.e.,

$$\text{Höld}_{\alpha,\Omega}(f) := \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

If f is Lipschitz, i.e., $f \in C^{0,1}(\overline{\Omega})$, we use

$$\text{Lip}_\Omega(f) = \text{Höld}_{1,\Omega}(f).$$

We define

$$C^{k,\alpha}(\overline{\Omega}) = \{f \in C^k(\overline{\Omega}) : \partial^\beta f \in C^{0,\alpha}(\overline{\Omega}), |\beta| = k\}.$$

Note that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space when provided with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} := \sup_{\substack{|\beta| \leq k \\ x \in \Omega}} |\partial^\beta f(x)| + \sup_{|\beta|=k} \text{Höld}_{\alpha,\Omega}(\partial^\beta f).$$

2.2. Lebesgue spaces and weak Lebesgue spaces. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $1 \leq p \leq \infty$. We denote by $L^p(\Omega)$ the Lebesgue space with respect to the n -dimensional Lebesgue measure \mathcal{L}^n . For Lebesgue measurable sets $E \subseteq \mathbb{R}^n$ we will denote by

$$|E| = \mathcal{L}^n(E)$$

its n -dimensional Lebesgue measure. We denote by p' the conjugate exponent of p defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

with the convention $1' = \infty$ and $\infty' = 1$.

Let $1 \leq p < \infty$. A measurable function $f : \Omega \rightarrow \mathbb{C}$ belongs to the weak L^p -space $L_w^p(\Omega)$ if

$$\|f\|_{p,w,\Omega} := \sup_{r \geq 0} r |\{x \in \Omega : |f(x)| > r\}|^{1/p} < \infty.$$

For $1 \leq q < p < \infty$ we have (cf. [17, Ex. 1.1.11])

$$(2.1) \quad \|f\|_{q,w,\Omega} \leq \|f\|_{L^q(\Omega)} \leq \left(\frac{p}{p-q}\right)^{1/q} |\Omega|^{1/q-1/p} \|f\|_{p,w,\Omega}$$

and hence $L^p(\Omega) \subseteq L_w^p(\Omega) \subseteq L^q(\Omega) \subseteq L_w^q(\Omega)$ with strict inclusions. It will be convenient to *normalize* the L^p -norm and the L_w^p -quasinorm, i.e., we will consider

$$\begin{aligned} \|f\|_{L^p(\Omega)}^* &:= |\Omega|^{-1/p} \|f\|_{L^p(\Omega)}, \\ \|f\|_{p,w,\Omega}^* &:= |\Omega|^{-1/p} \|f\|_{p,w,\Omega}. \end{aligned}$$

Note that $\|1\|_{L^p(\Omega)}^* = \|1\|_{p,w,\Omega}^* = 1$. Then, for $1 \leq q < p < \infty$,

$$(2.2) \quad \|f\|_{L^q(\Omega)}^* \leq \|f\|_{L^p(\Omega)}^*,$$

$$(2.3) \quad \|f\|_{q,w,\Omega}^* \leq \|f\|_{L^q(\Omega)}^* \leq \left(\frac{p}{p-q}\right)^{1/q} \|f\|_{p,w,\Omega}^*.$$

We remark that $\|\cdot\|_{p,w,\Omega}$ is only a quasinorm; the triangle inequality fails, but for $f_j \in L_w^p(\Omega)$ we still have

$$\left\| \sum_{j=1}^m f_j \right\|_{p,w,\Omega} \leq m \sum_{j=1}^m \|f_j\|_{p,w,\Omega}.$$

There exists a norm equivalent to $\|\cdot\|_{p,w,\Omega}$ which makes $L_w^p(\Omega)$ into a Banach space if $p > 1$.

The L_w^p -quasinorm is σ -subadditive: if $\{\Omega_j\}$ is a countable family of open sets with $\Omega = \bigcup \Omega_j$ then

$$(2.4) \quad \|f\|_{p,w,\Omega}^p \leq \sum_j \|f\|_{p,w,\Omega_j}^p \quad \text{for every } f \in L_w^p(\Omega).$$

But it is not σ -additive: for instance, for $h : (0, \infty) \rightarrow \mathbb{R}$, $h(t) := t^{-1/p}$, we have $\|h\|_{p,w,(0,\epsilon)}^p = 1$ for every $\epsilon > 0$, but $\|h\|_{p,w,(1,2)}^p = 1/2$.

2.3. Sobolev spaces. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we consider the Sobolev space

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega), 0 \leq |\alpha| \leq k\},$$

where $\partial^\alpha f$ denote distributional derivatives. On bounded intervals $I \subseteq \mathbb{R}$ the Sobolev space $W^{1,1}(I)$ coincides with the space $AC(I)$ of absolutely continuous functions on I if we identify each $W^{1,1}$ -functions with its unique continuous representative. Recall that a function $f : \Omega \rightarrow \mathbb{R}$ on an open subset $\Omega \subseteq \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$

there exists $\delta > 0$ so that $\sum_{i=1}^n |a_i - b_i| < \delta$ implies $\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon$ whenever $[a_i, b_i]$, $i = 1, \dots, n$, are non-overlapping intervals contained in Ω .

We shall also use $W_{\text{loc}}^{k,p}$, AC_{loc} , etc. with the obvious meaning.

2.4. Extension lemma. We will use the following extension lemma. The analogue for L_w^p -quasinorms may be found in [29, Lemma 2.1] which is a slight generalization of [15, Lemma 3.2]. Here we need a version for L^p -norms.

Lemma 1. *Let $\Omega \subseteq \mathbb{R}$ be open and bounded, let $f : \Omega \rightarrow \mathbb{C}$ be continuous, and set $\Omega_0 := \{t \in \Omega : f(t) \neq 0\}$. Assume that $f|_{\Omega_0} \in AC_{\text{loc}}(\Omega_0)$ and that $f'|_{\Omega_0} \in L^p(\Omega_0)$ for some $p > 1$ (note that f is differentiable a.e. in Ω_0). Then the distributional derivative of f in Ω is a measurable function $f' \in L^p(\Omega)$ and*

$$(2.5) \quad \|f'\|_{L^p(\Omega)} = \|f'|_{\Omega_0}\|_{L^p(\Omega_0)}.$$

Proof. The function $\psi : \Omega \rightarrow \mathbb{C}$ defined by

$$\psi(t) := \begin{cases} f'(t) & \text{if } t \in \Omega_0 \\ 0 & \text{if } t \in \Omega \setminus \Omega_0 \end{cases}$$

clearly belongs to $L^p(\Omega)$. We show that ψ is the distributional derivative of f in Ω . Let $\phi \in C_c^\infty(\Omega)$ be a test function with compact support in Ω and let \mathcal{C} denote the (at most countable) set of connected components of Ω_0 . Then, using integration by parts for the Lebesgue integral (see e.g. [25] Corollary 3.37)

$$\int_{\Omega} f \phi' dt = \int_{\Omega_0} f \phi' dt = \sum_{J \in \mathcal{C}} \int_J f \phi' dt = - \sum_{J \in \mathcal{C}} \int_J f' \phi dt = - \int_{\Omega_0} f' \phi dt = - \int_{\Omega} \psi \phi dt.$$

(If $J = (a, b)$ then $\int_a^b f \phi' dt = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} f \phi' dt = - \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} f' \phi dt = - \int_a^b f' \phi dt$, by the dominated convergence theorem, continuity of f , and (2.2).) Moreover, we have $\|f'\|_{L^p(\Omega)} = \|\psi\|_{L^p(\Omega)} = \|\psi\|_{L^p(\Omega_0)} = \|f'|_{\Omega_0}\|_{L^p(\Omega_0)}$, that is (2.5). \square

3. RADICALS OF DIFFERENTIABLE FUNCTIONS

We derive an analogue of Theorem 3 for complex valued functions.

Proposition 1. *Let $I \subseteq \mathbb{R}$ be a bounded interval, let $k \in \mathbb{N}_{>0}$, and $\alpha \in (0, 1]$. For each $g \in C^{k,\alpha}(\bar{I})$ we have*

$$(3.1) \quad |g'(t)| \leq \Lambda_{k+\alpha}(t) |g(t)|^{1-1/(k+\alpha)}, \quad \text{a.e. in } I,$$

for some $\Lambda_{k+\alpha} = \Lambda_{k+\alpha,g} \in L_w^p(I, \mathbb{R}_{\geq 0})$, where $1/p + 1/(k+\alpha) = 1$, and such that

$$(3.2) \quad \|\Lambda_{k+\alpha}\|_{p,w,I} \leq C(k) \max \left\{ \left(\text{Höld}_{\alpha,I}(g^{(k)}) \right)^{1/(k+\alpha)} |I|^{1/p}, \|g'\|_{L^\infty(I)}^{1/(k+\alpha)} \right\}.$$

Proof. Analogous to the proof of [29, Proposition 3.1]. \square

Corollary 1. *Let n be a positive integer and let $I \subseteq \mathbb{R}$ be an open bounded interval. Assume that $f : I \rightarrow \mathbb{C}$ is a continuous function such that $f^n = g \in C^{n-1,1}(\bar{I})$. Then we have $f' \in L_w^{n'}(I)$ and*

$$(3.3) \quad \|f'\|_{n',w,I} \leq C(n) \max \left\{ \left(\text{Lip}_I(g^{(n-1)}) \right)^{1/n} |I|^{1/n'}, \|g'\|_{L^\infty(I)}^{1/n} \right\},$$

where $C(n)$ is a constant that depends only on n and $1/n + 1/n' = 1$.

Proof. On the set $\Omega_0 = \{t \in I : f(t) \neq 0\}$, f is differentiable and satisfies

$$|f'(t)| = \frac{1}{n} \frac{|g'(t)|}{|g(t)|^{1-1/n}}.$$

So the assertion follows from Proposition 1 and the L_w^p -analogue of Lemma 1; see [29, Lemma 2.1]. \square

Remark 1. Proposition 1 and hence also Corollary 1 are optimal in the following sense:

- $\Lambda_{k+\alpha}$ can in general not be chosen in L^p . Indeed, for $g : (-1, 1) \rightarrow \mathbb{R}$, $g(t) = t$, we have

$$\left(\frac{|g'|}{|g|^{1-1/(k+\alpha)}} \right)^p = |t|^{-1},$$

which is not integrable near 0. See [15, Example 4.3].

- If f is just in $C^{k,\beta}(\bar{I})$ for every $\beta < \alpha$, then (3.1) does in general not hold with $\Lambda_{k+\alpha} \in L^1(I)$. Indeed, in [15, Example 4.4] there is constructed a non-negative function $f : I \rightarrow \mathbb{R}$ which belongs to $C^{k,\beta}(\bar{I}) \cap C^\infty(I)$ for every $\beta < \alpha$, but not for $\beta = \alpha$, and whose non-negative $(k + \alpha)$ -root has unbounded variation in I .

4. PRELIMINARIES ON POLYNOMIALS

4.1. **Tschirnhausen transformation.** A monic polynomial

$$P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}, \quad a = (a_1, \dots, a_n) \in \mathbb{C}^n$$

is said to be in *Tschirnhausen form* if $a_1 = 0$. Every polynomial P_a can be transformed to a polynomial $P_{\tilde{a}}$ in Tschirnhausen form by the substitution $Z \mapsto Z - a_1/n$, which we refer to as the *Tschirnhausen transformation*,

$$P_{\tilde{a}}(Z) = P_a(Z - a_1/n) = Z^n + \sum_{j=2}^n \tilde{a}_j Z^{n-j}, \quad \tilde{a} = (\tilde{a}_2, \dots, \tilde{a}_n) \in \mathbb{C}^{n-1}.$$

We have the formulas

$$(4.1) \quad \tilde{a}_j = \sum_{\ell=0}^j C_\ell a_\ell a_1^{j-\ell}, \quad j = 2, \dots, n,$$

where C_ℓ are universal constants. The effect of the Tschirnhausen transformation will always be indicated by adding *tilde* to the coefficients, $P_a \rightsquigarrow P_{\tilde{a}}$.

We will identify the set of monic complex polynomials P_a of degree n with the set \mathbb{C}^n (via $P_a \mapsto a$) and the set of monic complex polynomials $P_{\tilde{a}}$ of degree n in Tschirnhausen form with the set \mathbb{C}^{n-1} (via $P_{\tilde{a}} \mapsto \tilde{a}$).

4.2. Splitting. The following well-known lemma (see e.g. [1] or [4]) is a consequence of the inverse function theorem.

Lemma 2. *Let $P_a = P_b P_c$, where P_b and P_c are monic complex polynomials without common root. Then for P near P_a we have $P = P_{b(P)} P_{c(P)}$ for analytic mappings of monic polynomials $P \mapsto b(P)$ and $P \mapsto c(P)$, defined for P near P_a , with the given initial values.*

Proof. The splitting $P_a = P_b P_c$ defines on the coefficients a polynomial mapping φ such that $a = \varphi(b, c)$, where $a = (a_i)$, $b = (b_i)$, and $c = (c_i)$. The Jacobian determinant $\det d\varphi(b, c)$ equals the resultant of P_b and P_c which is nonzero by assumption. Thus φ can be inverted locally. \square

If $P_{\tilde{a}}$ is in Tschirnhausen form and if $\tilde{a} \neq 0$, then $P_{\tilde{a}}$ splits, i.e., $P_{\tilde{a}} = P_b P_c$ for monic polynomials P_b and P_c with positive degree and without common zero. For, if $\lambda_1, \dots, \lambda_n$ denote the roots of $P_{\tilde{a}}$ and they all coincide, then since

$$\lambda_1 + \dots + \lambda_n = \tilde{a}_1 = 0$$

they all must vanish, contradicting $\tilde{a} \neq 0$.

Let us identify the set of monic complex polynomials $P_{\tilde{a}}$ of degree n in Tschirnhausen form with the set \mathbb{C}^{n-1} , and let $\tilde{a}_2, \dots, \tilde{a}_n$ denote the coordinates in \mathbb{C}^{n-1} . Fix $k \in \{2, \dots, n\}$ and let $p \in \mathbb{C}^{n-1} \cap \{\tilde{a}_k \neq 0\}$; p corresponds to the polynomial $P_{\tilde{a}}$. We associate the polynomial

$$Q_{\underline{a}}(Z) := \tilde{a}_k^{-n/k} P_{\tilde{a}}(\tilde{a}_k^{1/k} Z) = Z^n + \sum_{j=2}^n \tilde{a}_k^{-j/k} \tilde{a}_j Z^{n-j},$$

$$\underline{a}_j := \tilde{a}_k^{-j/k} \tilde{a}_j, \quad j = 2, \dots, n,$$

where the radicals are interpreted as multi-valued functions. Then $Q_{\underline{a}}$ is in Tschirnhausen form and $\underline{a}_k = 1$. By Lemma 2 we have a splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ on some open ball $B_\rho(p)$ centered at p with radius $\rho > 0$. In particular, there exist analytic functions ψ_i so that, on $B_\rho(p)$,

$$\underline{b}_i = \psi_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \tilde{a}_k^{-3/k} \tilde{a}_3, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 1, \dots, \deg P_{\underline{b}}.$$

The splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ induces a splitting $P_{\tilde{a}} = P_b P_c$, where

$$(4.2) \quad b_i = \tilde{a}_k^{i/k} \psi_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \tilde{a}_k^{-3/k} \tilde{a}_3, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 1, \dots, \deg P_b;$$

likewise for c_j . Shrinking ρ slightly, we may assume that all partial derivatives of ψ_i are separately bounded on $B_\rho(p)$. If \tilde{b}_j denote the coefficients of the polynomial $P_{\tilde{b}}$ resulting from P_b by the Tschirnhausen transformation, then, by (4.1),

$$(4.3) \quad \tilde{b}_i = \tilde{a}_k^{i/k} \tilde{\psi}_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \tilde{a}_k^{-3/k} \tilde{a}_3, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 2, \dots, \deg P_b;$$

for analytic functions $\tilde{\psi}_i$ all partial derivatives of which are separately bounded on $B_\rho(p)$.

4.3. Coefficient estimates. We shall need the following estimates. (Here it is convenient to number the coefficients in reversed order.)

Lemma 3. *Let $m \geq 1$ be an integer and $\alpha \in (0, 1]$. Let $P(x) = a_1x + \cdots + a_mx^m \in \mathbb{C}[x]$ satisfy*

$$(4.4) \quad |P(x)| \leq A(1 + Mx^{m+\alpha}), \quad \text{for } x \in [0, B] \subseteq \mathbb{R},$$

and constants $A, M \geq 0$ and $B > 0$. Then

$$(4.5) \quad |a_j| \leq CA(1 + M^{j/(m+\alpha)}B^j)B^{-j}, \quad j = 1, \dots, m,$$

for a constant C depending only on m and α .

Proof. The statement is well-known if $M = 0$; see [31, Lemma 3.4]. Assume that $M > 0$.

It suffices to consider the special case $A = B = 1$. The general case follows by applying the special case to $Q(x) = A^{-1}P(Bx) = b_1x + \cdots + b_mx^m$, where $b_i = A^{-1}B^i a_i$.

Fix $k \in \{1, \dots, m\}$ and write the inequality (4.4) in the form

$$(4.6) \quad |x^{-k}P(x)| \leq x^{-k} + Mx^{m+\alpha-k}.$$

The function on the right-hand side of (4.6) attains its minimum on $\{x > 0\}$ at the point

$$(4.7) \quad x_k = \left(\frac{k}{m+\alpha-k}\right)^{1/(m+\alpha)} M^{-1/(m+\alpha)}$$

and this minimum is of the form $C_k M^{k/(m+\alpha)}$ for some C_k depending only on k, m , and α . Thus,

$$(4.8) \quad |P(x_k)| \leq \tilde{C}_k,$$

for some \tilde{C}_k depending only on k, m , and α .

Suppose first that $x_k \leq 1$ for all $k = 1, \dots, m$ and consider

$$a_1x_k + \cdots + a_mx_k^m = P(x_k), \quad k = 1, \dots, m,$$

as a system of linear equations with the unknowns $a_k M^{-k/(m+\alpha)}$ and the (Vandermonde-like) matrix

$$L = \left(\left(\frac{k}{m+\alpha-k}\right)^{j/(m+\alpha)}\right)_{k,j=1}^m.$$

Then the vector of unknowns is given by

$$(a_1 M^{-1/(m+\alpha)}, \dots, a_m M^{-m/(m+\alpha)})^T = L^{-1}(P(x_1), P(x_2), \dots, P(x_m))^T.$$

By (4.8), we may conclude that

$$|a_j| \leq CM^{j/(m+\alpha)}, \quad j = 1, \dots, m,$$

for a constant C depending only on m and α .

If $x_k > 1$ then $M < k/(m + \alpha - k)$, by (4.7), and hence for $x \in [0, 1]$,

$$|P(x)| \leq 1 + Mx^{m+\alpha} \leq 1 + \frac{k}{m+\alpha-k} \leq \frac{m+\alpha}{\alpha}.$$

In this case we may apply the lemma with $M = 0$, $A = (m + \alpha)/\alpha$, and $B = 1$, and obtain

$$|a_j| \leq C, \quad j = 1, \dots, m,$$

for a constant C depending only on m and α .

Summing up, we obtained (4.5) in the case $A = B = 1$. □

As a consequence we get estimates for the intermediate derivatives of a finitely differentiable function in terms of the function and its highest derivative. For an interval $I \subseteq \mathbb{R}$ and a function $f : I \rightarrow \mathbb{C}$ we define

$$V_I(f) := \sup_{t,s \in I} |f(t) - f(s)|.$$

Lemma 4. *Let $I \subseteq \mathbb{R}$ be a bounded open interval, $m \in \mathbb{N}$, and $\alpha \in (0, 1]$. If $f \in C^{m,\alpha}(\bar{I})$, then there is a universal constant C , depending only on m and α , such that for all $t \in I$ and $s = 1, \dots, m$,*

$$(4.9) \quad |f^{(s)}(t)| \leq C|I|^{-s} (V_I(f) + V_I(f)^{(m+\alpha-s)/(m+\alpha)} (\text{Höld}_{\alpha,I}(f^{(m)}))^{s/(m+\alpha)} |I|^s).$$

Proof. We may suppose that $I = (-\delta, \delta)$. If $t \in I$ then at least one of the two intervals $[t, t \pm \delta)$, say $[t, t + \delta)$, is included in I . By Taylor's formula, for $t_1 \in [t, t + \delta)$,

$$\sum_{s=1}^m \frac{f^{(s)}(t)}{s!} (t_1 - t)^s = f(t_1) - f(t) - \int_0^1 \frac{(1-\tau)^{m-1}}{(m-1)!} (f^{(m)}(t + \tau(t_1 - t)) - f^{(m)}(t)) d\tau (t_1 - t)^m$$

and hence

$$\begin{aligned} \left| \sum_{s=1}^m \frac{f^{(s)}(t)}{s!} (t_1 - t)^s \right| &\leq V_I(f) + \text{Höld}_{\alpha,I}(f^{(m)})(t_1 - t)^{m+\alpha} \\ &= V_I(f) (1 + V_I(f)^{-1} \text{Höld}_{\alpha,I}(f^{(m)})(t_1 - t)^{m+\alpha}). \end{aligned}$$

The assertion follows from Lemma 3. \square

4.4. Higher order Glaeser inequalities. As a corollary of Lemma 4 we obtain a generalization of Ghisi and Gobbino's higher order Glaeser inequalities [15, Prop. 3.4].

Corollary 2. *Let $m \in \mathbb{N}$ and $\alpha \in (0, 1]$. Let $I = (t_0 - \delta, t_0 + \delta)$ with $t_0 \in \mathbb{R}$ and $\delta > 0$. If $f \in C^{m,\alpha}(\bar{I})$ is such that f and f' do not change their sign on I , then there is a universal constant C , depending only on m and α , such that for all $s = 1, \dots, m$,*

$$(4.10) \quad |f^{(s)}(t_0)| \leq C|I|^{-s} (|f(t_0)| + |f(t_0)|^{(m+\alpha-s)/(m+\alpha)} (\text{Höld}_{\alpha,I}(f^{(m)}))^{s/(m+\alpha)} |I|^s).$$

Proof. For simplicity assume $t_0 = 0$. Changing f to $-f$ and t to $-t$ if necessary, we may assume that $f(t) \geq 0$ and $f'(t) \leq 0$ for all $t \geq 0$. Then $V_{[0,\delta)}(f) \leq f(0)$ and so (4.10) follows from (4.9). \square

For $s = 1$ we recover [15, Prop. 3.4]. Indeed, for $s = 1$ we may write (4.10) in the form

$$(4.11) \quad |f'(t_0)| \leq C|f(t_0)|^{(m+\alpha-1)/(m+\alpha)} \max\{|f(t_0)|^{1/(m+\alpha)} |I|^{-1}, (\text{Höld}_{\alpha,I}(f^{(m)}))^{1/(m+\alpha)}\},$$

and the inequality in [15, Prop. 3.4] can be written as

$$(4.12) \quad |f'(t_0)| \leq C|f(t_0)|^{(m+\alpha-1)/(m+\alpha)} \max\{|f'(t_0)|^{1/(m+\alpha)} |I|^{-1+1/(m+\alpha)}, (\text{Höld}_{\alpha,I}(f^{(m)}))^{1/(m+\alpha)}\}.$$

These two inequalities are equivalent in the following sense. If (4.11) holds with the constant $C > 0$ then (4.12) holds with the constant $\max\{C, C^{(m+\alpha-1)/(m+\alpha)}\}$ and symmetrically, if (4.12) holds with the constant $C > 0$ then (4.11) holds with the constant $\max\{C, C^{(m+\alpha)/(m+\alpha-1)}\}$. For instance, suppose that (4.11) holds. If the second term in

the maximum is dominant then (4.12) holds with the same constant. If the first term is dominant in the maximum, that is $|f'(t_0)| \leq C|f(t_0)||I|^{-1}$, then $|f'(t_0)|^{(m+\alpha-1)/(m+\alpha)} \leq (C|f(t_0)||I|^{-1})^{(m+\alpha-1)/(m+\alpha)}$ and (4.12) holds with the constant $C^{(m+\alpha-1)/(m+\alpha)}$.

5. ESTIMATES FOR THE ITERATED DERIVATIVES OF THE COEFFICIENTS

5.1. Preparations for the splitting. Let $I \subseteq \mathbb{R}$ be a bounded open interval and let

$$(5.1) \quad P_{\tilde{a}(t)}(Z) = Z^n + \sum_{j=2}^n \tilde{a}_j(t) Z^{n-j}, \quad t \in I,$$

be a monic complex polynomial in Tschirnhausen form with coefficients $\tilde{a}_j \in C^{n-1,1}(\bar{I})$, $j = 2, \dots, n$. We make the following assumptions. Suppose that $t_0 \in I$ and $k \in \{2, \dots, n\}$ are such that

$$(5.2) \quad |\tilde{a}_k(t_0)|^{1/k} = \max_{2 \leq j \leq n} |\tilde{a}_j(t_0)|^{1/j} \neq 0$$

and that, for some positive constant $B < 1/3$,

$$(5.3) \quad \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \leq B|\tilde{a}_k(t_0)|^{1/k}.$$

By Corollary 1, every continuous selection f of the multi-valued function $\tilde{a}_j^{1/j}$ is absolutely continuous on I and $\|f'\|_{L^1(I)}$ is independent of the choice of the selection, by (2.3) and (3.3). (By a selection of a set-valued function $F : X \rightsquigarrow Y$ we mean a single-valued function $f : X \rightarrow Y$ such that $f(x) \in F(x)$ for all $x \in X$.) So henceforth we shall fix one continuous selection of $\tilde{a}_j^{1/j}$, and, abusing notation, denote it by $\tilde{a}_j^{1/j}$ as well.

Lemma 5. *Assume that the polynomial (5.1) satisfies (5.2)–(5.3). Then for all $t \in I$ and $j = 2, \dots, n$,*

$$(5.4) \quad |\tilde{a}_j^{1/j}(t) - \tilde{a}_j^{1/j}(t_0)| \leq B|\tilde{a}_k(t_0)|^{1/k},$$

$$(5.5) \quad \frac{2}{3} < 1 - B \leq \left| \frac{\tilde{a}_k(t)}{\tilde{a}_k(t_0)} \right|^{1/k} \leq 1 + B < \frac{4}{3},$$

$$(5.6) \quad |\tilde{a}_j(t)|^{1/j} \leq \frac{4}{3}|\tilde{a}_k(t_0)|^{1/k} \leq 2|\tilde{a}_k(t)|^{1/k}.$$

Proof. In fact, by (5.3),

$$|\tilde{a}_j^{1/j}(t) - \tilde{a}_j^{1/j}(t_0)| = \left| \int_{t_0}^t (\tilde{a}_j^{1/j})' ds \right| \leq \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \leq B|\tilde{a}_k(t_0)|^{1/k},$$

that is (5.4). For $j = k$ it implies

$$\left| \frac{\tilde{a}_k(t)}{\tilde{a}_k(t_0)} \right|^{1/k} - 1 \leq B,$$

and thus (5.5). By (5.2), (5.4), and (5.5),

$$|\tilde{a}_j(t)|^{1/j} \leq (1 + B)|\tilde{a}_k(t_0)|^{1/k} \leq 2|\tilde{a}_k(t)|^{1/k},$$

that is (5.6). □

By (5.5), \tilde{a}_k does not vanish on the interval I and so the curve

$$(5.7) \quad \begin{aligned} \underline{a} : I &\rightarrow \{(\underline{a}_2, \dots, \underline{a}_n) \in \mathbb{C}^{n-1} : \underline{a}_k = 1\} \\ t &\mapsto \underline{a}(t) := (\tilde{a}_k^{-2/k} \tilde{a}_2, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n)(t) \end{aligned}$$

is well-defined.

Lemma 6. *Assume that the polynomial (5.1) satisfies (5.2)–(5.3). Then the length of the curve (5.7) is bounded by $3n^2 2^n B$.*

Proof. The estimates (5.4), (5.5), and (5.6) imply

$$\begin{aligned} |\tilde{a}_k^{-j/k} \tilde{a}_j'| &\leq 2^n |\tilde{a}_j^{-1+1/j} \tilde{a}_j' \tilde{a}_k^{-1/k}| \leq 3n 2^{n-1} |(\tilde{a}_j^{1/j})'| |\tilde{a}_k(t_0)|^{-1/k} \\ |(\tilde{a}_k^{-j/k})' \tilde{a}_j| &\leq n 2^n |\tilde{a}_k^{-1/k} (\tilde{a}_k^{1/k})'| \leq 3n 2^{n-1} |(\tilde{a}_k^{1/k})'| |\tilde{a}_k(t_0)|^{-1/k}, \end{aligned}$$

and thus

$$|(\tilde{a}_k^{-j/k} \tilde{a}_j')| \leq 3n 2^{n-1} |\tilde{a}_k(t_0)|^{-1/k} \left(|(\tilde{a}_j^{1/j})'| + |(\tilde{a}_k^{1/k})'| \right).$$

Consequently, using (5.3),

$$\int_I |\underline{a}'| ds \leq 3n^2 2^n B,$$

as required. □

5.2. Estimates for the derivatives of the coefficients. Let us replace (5.3) by the stronger assumption

$$(5.8) \quad M|I| + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \leq B |\tilde{a}_k(t_0)|^{1/k},$$

where

$$(5.9) \quad M = \max_{2 \leq j \leq n} (\text{Lip}_I(\tilde{a}_j^{(n-1)}))^{1/n} |\tilde{a}_k(t_0)|^{(n-j)/(kn)}.$$

Lemma 7. *Assume that the polynomial (5.1) satisfies (5.2) and (5.8). Then for all $t \in I$, $j = 2, \dots, n$, and $s = 1, \dots, n$,*

$$(5.10) \quad |\tilde{a}_j^{(s)}(t)| \leq C(n) |I|^{-s} |\tilde{a}_k(t_0)|^{j/k}.$$

Proof. By Lemma 4,

$$|\tilde{a}_j^{(s)}(t)| \leq C |I|^{-s} (V_I(\tilde{a}_j) + V_I(\tilde{a}_j)^{(n-s)/n} \text{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n} |I|^s).$$

By (5.6),

$$V_I(\tilde{a}_j) \leq 2 \|\tilde{a}_j\|_{L^\infty(I)} \leq 2 (4/3)^n |\tilde{a}_k(t_0)|^{j/k}$$

and, by (5.8),

$$\max_{2 \leq j \leq n} (\text{Lip}_I(\tilde{a}_j^{(n-1)}))^{s/n} |\tilde{a}_k(t_0)|^{-js/(kn)} |I|^s = |\tilde{a}_k(t_0)|^{-s/k} M^s |I|^s \leq 1.$$

Thus

$$\begin{aligned}
V_I(\tilde{a}_j) + V_I(\tilde{a}_j)^{(n-s)/n} \text{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n} |I|^s \\
\leq |\tilde{a}_k(t_0)|^{j/k} (C_1 + C_2 \text{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n} |\tilde{a}_k(t_0)|^{-js/(kn)} |I|^s) \\
\leq C_3 |\tilde{a}_k(t_0)|^{j/k},
\end{aligned}$$

for constants C_i that depend only on n . So (5.10) is proved. \square

6. THE ESTIMATES AFTER SPLITTING

6.1. Estimates after splitting on I . Assume that the polynomial (5.1) satisfies (5.2)-(5.3) and the estimates (5.10).

We suppose additionally that the curve \underline{a} as defined in (5.7) lies entirely in one of the balls $B_\rho(p)$ from Section 4.2 on which we have a splitting. Then $P_{\tilde{a}}$ splits on I ,

$$(6.1) \quad P_{\tilde{a}}(t) = P_b(t) P_{b^*}(t), \quad t \in I.$$

By (4.2) and (4.3), the coefficients b_i are of the form

$$(6.2) \quad b_i = \tilde{a}_k^{i/k} \psi_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 1, \dots, \deg P_b,$$

and after Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde{b}}$, we get

$$(6.3) \quad \tilde{b}_i = \tilde{a}_k^{i/k} \tilde{\psi}_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 2, \dots, \deg P_b,$$

where ψ_i and $\tilde{\psi}_i$ are the analytic functions specified in Section 4.2.

Lemma 8. *Assume that the polynomial (5.1) satisfies (5.2)-(5.3), (5.10), and (6.1)-(6.3). Then there is a universal constant C , depending only on n and on the functions $\tilde{\psi}_i$, such that for all $t \in I$, $i = 2, \dots, \deg P_b$, and $s = 1, \dots, n$,*

$$(6.4) \quad |\tilde{b}_i^{(s)}(t)| \leq C |I|^{-s} |\tilde{a}_k(t_0)|^{i/k}.$$

Moreover, for all $t \in I$,

$$(6.5) \quad |b'_1(t)| \leq C |I|^{-1} |\tilde{a}_k(t_0)|^{1/k}.$$

Proof. We claim that the functions $\tilde{\psi}_i \circ \underline{a}$ satisfy

$$(6.6) \quad |\partial_t^s(\tilde{\psi}_i \circ \underline{a})| \leq C |I|^{-s},$$

for a constant C as required in the lemma. Using induction, (5.10), and differentiating the following equation $(s-1)$ times,

$$\partial_t(\tilde{\psi}_i \circ \underline{a}) = \sum_{j=2}^n ((\partial_{j-1} \tilde{\psi}_i) \circ \underline{a}) \partial_t(\tilde{a}_k^{-j/k} \tilde{a}_j),$$

(6.6) follows easily. (Here we used the fact that all partial derivatives of the functions $\tilde{\psi}_i$ are separately bounded and that these bounds are universal.) Now (6.4) is a consequence of (6.3) and (6.6).

The proof of (6.5) is analogous. \square

6.2. Intervals of first and second kind. Assume that the polynomial (5.1) satisfies (5.2)–(5.3), (5.10), and (6.1)–(6.3). Suppose that $t_1 \in I$ and $\ell \in \{2, \dots, \deg P_b\}$ are such that

$$(6.7) \quad |\tilde{b}_\ell(t_1)|^{1/\ell} = \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(t_1)|^{1/i} \neq 0.$$

By (5.6) and (6.3), for all $t \in I$ and $i = 2, \dots, \deg P_b$,

$$(6.8) \quad |\tilde{b}_i(t)| \leq C_1 |\tilde{a}_k(t_0)|^{i/k},$$

where the constant C_1 depends only on the functions $\tilde{\psi}_i$. As an immediate consequence of (6.8) we may conclude that we can choose a universal constant $D < 1/3$ and that there is an open interval J , $t_1 \subseteq J \subseteq I$, such that

$$(6.9) \quad |J||I|^{-1} |\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J)} = D |\tilde{b}_\ell(t_1)|^{1/\ell}.$$

It suffices to choose $D < C_1^{-1}$ where C_1 is the constant in (6.8); $\tilde{b}_i^{1/i}$ is absolutely continuous by Corollary 1. Let us set

$$\begin{aligned} \varphi_{t_1,+}(s) &:= (s - t_1)|I|^{-1} |\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1([t_1, s])}, \quad s \geq t_1, \\ \varphi_{t_1,-}(s) &:= (t_1 - s)|I|^{-1} |\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((s, t_1])}, \quad s \leq t_1. \end{aligned}$$

Then $\varphi_{t_1,\pm} \geq 0$ are continuous with $\varphi_{t_1,\pm}(t_1) = 0$. We let $\varphi_{t_1,\pm}$ grow until $\varphi_{t_1,-}(s_-) + \varphi_{t_1,+}(s_+) = D |\tilde{b}_\ell(t_1)|^{1/\ell}$, that is (6.9) with $J = (s_-, s_+)$. And we do this *symmetrically* whenever possible:

(i) We say that the interval $J = (s_-, s_+)$ is of *first kind* if

$$(6.10) \quad \varphi_{t_1,-}(s_-) = \varphi_{t_1,+}(s_+) = \frac{D}{2} |\tilde{b}_\ell(t_1)|^{1/\ell}.$$

(ii) If (6.10) is not possible, i.e., we reach the boundary of the interval I before either $\varphi_{t_1,-}$ or $\varphi_{t_1,+}$ has grown to the value $(D/2) |\tilde{b}_\ell(t_1)|^{1/\ell}$, then we say that $J = (s_-, s_+)$ is of *second kind*.

6.3. Estimates after splitting on J .

Lemma 9. Assume that the polynomial (5.1) satisfies (5.2)–(5.3), (5.10), (6.1)–(6.3), and (6.7). Let D and J be as in (6.9). Then the functions \tilde{b}_i on J satisfy the conclusions of Lemmas 5, 6, and 7. More precisely, for all $t \in J$ and $i = 2, \dots, \deg P_b$,

$$(6.11) \quad |\tilde{b}_i^{1/i}(t) - \tilde{b}_i^{1/i}(t_1)| \leq D |\tilde{b}_\ell(t_1)|^{1/\ell},$$

$$(6.12) \quad \frac{2}{3} < 1 - D \leq \left| \frac{\tilde{b}_\ell(t)}{\tilde{b}_\ell(t_1)} \right|^{1/\ell} \leq 1 + D < \frac{4}{3},$$

$$(6.13) \quad |\tilde{b}_i(t)|^{1/i} \leq \frac{4}{3} |\tilde{b}_\ell(t_1)|^{1/\ell} \leq 2 |\tilde{b}_\ell(t)|^{1/\ell}.$$

The length of the curve

$$(6.14) \quad J \ni t \mapsto \underline{b}(t) := (\tilde{b}_\ell^{-2/\ell} \tilde{b}_2, \dots, \tilde{b}_\ell^{-\deg P_b/\ell} \tilde{b}_{\deg P_b})(t)$$

is bounded by $3(\deg P_b)^2 2^{\deg P_b} D$. There is a universal constant C , depending only on n and $\tilde{\psi}_i$, such that for all $t \in J$, $i = 2, \dots, \deg P_b$, and $s = 1, \dots, n$,

$$(6.15) \quad |\tilde{b}_i^{(s)}(t)| \leq C|J|^{-s} |\tilde{b}_\ell(t_1)|^{i/\ell}.$$

Proof. The proof of (6.11)–(6.13) is analogous to the proof of Lemma 5; use (6.7) and (6.9) instead of (5.2) and (5.3). The bound for the length of the curve $J \ni t \mapsto \underline{b}(t)$ (which is well-defined by (6.12)) follows from (6.9) and (6.11)–(6.13); see the proof of Lemma 6.

Let us prove (6.15). By (6.4), for $t \in I$ and $i = 2, \dots, \deg P_b$,

$$(6.16) \quad |\tilde{b}_i^{(i)}(t)| \leq C|I|^{-i} |\tilde{a}_k(t_0)|^{i/k},$$

where $C = C(n, \tilde{\psi}_i)$. Thus, for $t \in J$ and $s = 1, \dots, i$,

$$\begin{aligned} |\tilde{b}_i^{(s)}(t)| &\leq C|J|^{-s} (V_J(\tilde{b}_i) + V_J(\tilde{b}_i)^{(i-s)/i} \|\tilde{b}_i^{(i)}\|_{L^\infty(J)}^{s/i} |J|^s) && \text{by Lemma 4} \\ &\leq C_1|J|^{-s} \left(|\tilde{b}_\ell(t_1)|^{i/\ell} + |\tilde{b}_\ell(t_1)|^{(i-s)/\ell} |J|^s |I|^{-s} |\tilde{a}_k(t_0)|^{s/k} \right) && \text{by (6.13) and (6.16)} \\ &\leq C_2|J|^{-s} |\tilde{b}_\ell(t_1)|^{i/\ell} && \text{by (6.9),} \end{aligned}$$

for constants $C = C(i)$ and $C_i = C_i(n, \tilde{\psi}_i)$. For $s > i$, (6.4) implies

$$\begin{aligned} |\tilde{b}_i^{(s)}(t)| &\leq C|I|^{-s} |\tilde{a}_k(t_0)|^{i/k} = C|J|^{-s} (|J||I|^{-1})^s |\tilde{a}_k(t_0)|^{i/k} \\ &\leq C|J|^{-s} (|J||I|^{-1} |\tilde{a}_k(t_0)|^{1/k})^i \leq C|J|^{-s} |\tilde{b}_\ell(t_1)|^{i/\ell}, \end{aligned}$$

where the last inequality follows from (6.9). Thus (6.15) is proved. \square

7. A SPECIAL COVER BY INTERVALS

Assume that the polynomial (5.1) satisfies (5.2)–(5.3), (5.10), and (6.1)–(6.3). The arguments in Section 6.2 show that for each point t_1 in

$$I' := I \setminus \{t \in I : \tilde{b}_2(t) = \dots = \tilde{b}_{\deg P_b}(t) = 0\}$$

there exists $\ell \in \{2, \dots, \deg P_b\}$ such that (6.7) and there is an open interval $J = J(t_1)$, $t_1 \in J \subseteq I'$, such that (6.9); that $J \subseteq I'$ follows from (6.12).

The goal of this section is to prove the following proposition.

Proposition 2. *The collection $\{J(t_1)\}_{t_1 \in I'}$ has a countable subcollection \mathcal{J} that still covers I' and such that every point in I' belongs to at most two intervals in \mathcal{J} . In particular,*

$$\sum_{J \in \mathcal{J}} |J| \leq 2|I'|.$$

Remark 2. It is essential for us that \mathcal{J} is a subcollection and not a refinement; by shrinking the intervals we would lose equality in (6.9).

We can treat the connected components of I' separately. So let (α, β) be any connected component of I' and let $\mathcal{I} := \{J(t_1)\}_{t_1 \in (\alpha, \beta)}$. The coefficients \tilde{b}_i may or may not all vanish at the endpoints. We distinguish three cases:

(i) \tilde{b} vanishes at both endpoints,

$$(7.1) \quad \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\alpha)|^{1/i} = \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\beta)|^{1/i} = 0.$$

(ii) \tilde{b} vanishes at one endpoint, say α , but not at the other,

$$(7.2) \quad \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\alpha)|^{1/i} = 0, \quad \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\beta)|^{1/i} \neq 0.$$

(iii) \tilde{b} does not vanish at either endpoint,

$$(7.3) \quad \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\alpha)|^{1/i} \neq 0, \quad \sum_{i=2}^{\deg P_b} |\tilde{b}_i(\beta)|^{1/i} \neq 0.$$

Lemma 10. *We have:*

- (1) *If $\tilde{b}(\alpha) = 0$, then no interval $J \in \mathcal{I}$ has left endpoint α and $|J(t_1)| \rightarrow 0$ as $t_1 \rightarrow \alpha$. If $\tilde{b}(\beta) = 0$, then no interval $J \in \mathcal{I}$ has right endpoint β and $|J(t_1)| \rightarrow 0$ as $t_1 \rightarrow \beta$.*
- (2) *If $\tilde{b}(\alpha) \neq 0$, then there exists an interval $J \in \mathcal{I}$ of second kind (with endpoint α). If $\tilde{b}(\beta) \neq 0$, then there exists an interval $J \in \mathcal{I}$ of second kind (with endpoint β).*

Proof. (1) By (6.12), \tilde{b} cannot vanish at the endpoints of J . That $|J(t_1)| \rightarrow 0$ as t_1 tends to an endpoint, where \tilde{b} vanishes, is immediate from (6.9).

(2) Suppose that $\tilde{b}(\beta) \neq 0$. If all intervals $J(t_1)$ in \mathcal{I} were of first kind then, by (6.9) and (6.10),

$$(7.4) \quad \varphi_{t_1,+}(\beta) \geq \frac{D}{2} |\tilde{b}_{\ell}(t_1)|^{1/\ell} = \frac{D}{2} \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(t_1)|^{1/i}, \quad t_1 \in (\alpha, \beta).$$

But $\varphi_{t_1,+}(\beta) \rightarrow 0$ as $t_1 \rightarrow \beta$, while the right-hand side of (7.4) tends to a positive constant, a contradiction. \square

Lemma 11. *Suppose that $J \in \mathcal{I}$ and $t_1 \notin J$ such that $J(t_1)$ is of first kind. Then $J \not\subseteq J(t_1)$.*

Proof. Let $J = J(s_1) = (\alpha_{s_1}, \beta_{s_1})$ and assume without loss of generality that $\beta_{s_1} \leq t_1$. Suppose that $J(s_1) \subseteq J(t_1)$. Since $J(t_1) = (\alpha_{t_1}, \beta_{t_1})$ is of first kind (cf. (6.10)), we have

$$(t_1 - \alpha_{t_1}) |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((\alpha_{t_1}, t_1))} = \frac{D}{2} |\tilde{b}_{\ell_{t_1}}(t_1)|^{1/\ell_{t_1}} < D |\tilde{b}_{\ell_{s_1}}(s_1)|^{1/\ell_{s_1}},$$

because by (6.12) and (6.13),

$$|\tilde{b}_{\ell_{t_1}}(t_1)|^{1/\ell_{t_1}} < \frac{3}{2} |\tilde{b}_{\ell_{t_1}}(s_1)|^{1/\ell_{t_1}} \leq 2 |\tilde{b}_{\ell_{s_1}}(s_1)|^{1/\ell_{s_1}}.$$

But this leads to a contradiction in view of (6.9). \square

Case (i). By (7.1) and Lemma 10, each $J \in \mathcal{I}$ is an interval of first kind.

Choose any interval $J(t_1)$, $t_1 \in (\alpha, \beta)$, and denote it by $J_0 = (\alpha_0, \beta_0)$. Define recursively (for $\gamma \in \mathbb{Z}$)

$$J_\gamma = (\alpha_\gamma, \beta_\gamma) := \begin{cases} J(\beta_{\gamma-1}) & \text{if } \gamma \geq 1, \\ J(\alpha_{\gamma+1}) & \text{if } \gamma \leq -1. \end{cases}$$

By Lemma 11, we have $\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $\beta_\gamma < \beta_{\gamma+1} < \beta$ for all γ . Let us show that the collection $\mathcal{J} = \{J_\gamma\}_{\gamma \in \mathbb{Z}}$ covers (α, β) . Suppose that, say, $\tau := \sup_\gamma \beta_\gamma < \beta$. By (6.9) and since all intervals are of first kind (cf. (6.10)),

$$(\tau - \beta_\gamma)|I|^{-1}|\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((\beta_\gamma, \tau])} \geq \frac{D}{2} \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(\beta_\gamma)|^{1/i}.$$

But the left-hand side tends to 0 as $\gamma \rightarrow +\infty$, whereas the right-hand side converges to $(D/2) \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(\tau)|^{1/i} > 0$, a contradiction.

Now Proposition 2 follows from Lemma 10 and the following lemma.

Lemma 12. *Let $\mathcal{J} = \{J_\gamma\}_{\gamma \in \mathbb{Z}}$ be a countable collection of bounded open intervals $J_\gamma = (\alpha_\gamma, \beta_\gamma) \subseteq \mathbb{R}$ such that*

- (1) $\bigcup \mathcal{J} = (\alpha, \beta)$ is a bounded open interval,
- (2) $\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $\beta_\gamma < \beta_{\gamma+1} < \beta$ for all $\gamma \in \mathbb{Z}$,
- (3) $|J_\gamma| \rightarrow 0$ as $\gamma \rightarrow \pm\infty$.

Then there is a subcollection $\mathcal{J}_0 \subseteq \mathcal{J}$ with $\bigcup \mathcal{J}_0 = (\alpha, \beta)$ and such that every point in (α, β) belongs to at most two intervals in \mathcal{J}_0 .

Proof. The assumptions imply that the sequence of left endpoints (α_γ) converges to β as $\gamma \rightarrow \infty$, and the sequence of right endpoints (β_γ) converges to α as $\gamma \rightarrow -\infty$. Thus, there exists $\gamma_1 > 0$ such that $\alpha_{\gamma_1} < \beta_0 \leq \alpha_{\gamma_1+1}$, there exists $\gamma_2 > \gamma_1$ such that $\alpha_{\gamma_2} < \beta_{\gamma_1} \leq \alpha_{\gamma_2+1}$, and iteratively, there exists $\gamma_j > \gamma_{j-1}$ such that $\alpha_{\gamma_j} < \beta_{\gamma_{j-1}} \leq \alpha_{\gamma_j+1}$. Symmetrically, there exist integers $\gamma_{j-1} < \gamma_j < 0$ ($j \in \mathbb{Z}_{<0}$) such that $\beta_{\gamma_{j-1}-1} \leq \alpha_{\gamma_j} < \beta_{\gamma_{j-1}}$. Set $\gamma_0 := 0$ and define

$$\mathcal{J}_0 := \{J_{\gamma_j}\}_{j \in \mathbb{Z}}.$$

By construction \mathcal{J}_0 still covers (α, β) and the left and right endpoints of the intervals J_{γ_j} are interlacing,

$$\cdots < \beta_{\gamma_{j-2}} < \alpha_{\gamma_j} < \beta_{\gamma_{j-1}} < \alpha_{\gamma_{j+1}} < \beta_{\gamma_j} < \alpha_{\gamma_{j+2}} < \cdots$$

Thus \mathcal{J}_0 has the required properties. \square

Case (ii). By (7.2) and Lemma 10, the collection \mathcal{I} contains an interval of second kind. Since $\tilde{b}(\alpha) = 0$, all intervals of second kind in \mathcal{I} must have endpoint β . Thus, and because $|J(t_1)| \rightarrow 0$ as $t \rightarrow \alpha$ by Lemma 10,

$$\tau := \inf\{t_1 : J(t_1) \in \mathcal{I} \text{ is of second kind}\} > \alpha.$$

The interval $J(\tau)$ is of first kind (being of second kind is an open condition). There is an interval $J_0 = (\alpha_0, \beta_0 = \beta)$ of second kind in \mathcal{I} with $J(\tau) \cap J_0 \neq \emptyset$. Let us denote $J(\tau)$ by

$J_{-1} = (\alpha_{-1}, \beta_{-1})$ and define recursively

$$J_\gamma = (\alpha_\gamma, \beta_\gamma) := J(\alpha_{\gamma+1}), \quad \gamma \leq -1.$$

The arguments in Case (i) imply that the collection $\mathcal{J} := \{J_\gamma\}_{\gamma \leq 0}$ is a countable cover of (α, β) satisfying $\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $|J_\gamma| \rightarrow 0$.

Proposition 2 follows from (an obvious modification of) Lemma 12.

Case (iii). In this case \mathcal{I} has a finite subcollection \mathcal{J} that still covers (α, β) . Indeed, by (7.3) and Lemma 10, the collection \mathcal{I} contains intervals of second kind with endpoints α and β , say, (α, δ) and (ϵ, β) . If their intersection is non-empty we are done. Otherwise there are finitely many intervals in \mathcal{I} that cover the compact interval $[\delta, \epsilon]$.

Proposition 2 follows from the following lemma.

Lemma 13. *Every finite collection \mathcal{J} of open intervals with $\bigcup \mathcal{J} = (\alpha, \beta)$ has a subcollection that still covers (α, β) and every point in (α, β) belongs to at most two intervals in the subcollection.*

Proof. The collection \mathcal{J} contains an interval with endpoint α ; let $J_0 = (\alpha = \alpha_0, \beta_0)$ be the biggest among them. If $\beta_0 < \beta$, let $J_1 = (\alpha_1, \beta_1)$ denote the interval among all intervals in \mathcal{J} containing β_0 whose right endpoint is maximal. If $\beta_1 < \beta$, let $J_2 = (\alpha_2, \beta_2)$ denote the interval among all intervals in \mathcal{J} containing β_0 whose right endpoint is maximal, etc. This yields a finite cover of (α, β) by intervals $J_i = (\alpha_i, \beta_i)$, $i = 0, 1, \dots, N$, such that $\alpha_0 < \alpha_1 < \dots < \alpha_N$. Define

$$i_1 := \max_{\alpha_i < \beta_0} i, \quad i_j := \max_{\alpha_i < \beta_{i_{j-1}}} i, \quad j \geq 2.$$

Then $\{J_0, J_{i_1}, J_{i_2}, \dots, J_N\}$ has the required properties. \square

8. PROOF OF THEOREM 1

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$(8.1) \quad P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}, \quad t \in (\alpha, \beta),$$

be a monic polynomial with coefficients $a_j \in C^{n-1,1}([\alpha, \beta])$, $j = 1, \dots, n$.

The proof of Theorem 1 is by induction on the degree of the polynomial. We will reduce the degree by splitting P_a locally. Since the first splitting is atypical we shall consider subsequent splittings before we apply the inductive hypothesis.

8.1. Reduction to Tschirnhausen form. Without loss of generality we may assume that $n \geq 2$ and that $P_a = P_{\tilde{a}}$ is in Tschirnhausen form, i.e., $\tilde{a}_1 = 0$. We shall see in Section 8.8 how to get the bound (1.2) from a corresponding bound involving the \tilde{a}_j .

Let $\{\lambda_j(t)\}_{j=1}^n$, $t \in (\alpha, \beta)$, be any system of the roots of $P_{\tilde{a}}$ (not necessarily continuous). Since $P_{\tilde{a}}$ is in Tschirnhausen form, for fixed $t \in (\alpha, \beta)$,

$$(8.2) \quad \forall_{i,j} \lambda_i(t) = \lambda_j(t) \iff \forall_i \lambda_i(t) = 0 \iff \forall_i \tilde{a}_i(t) = 0.$$

8.2. Universal splitting of $P_{\tilde{a}}$. The space of monic polynomials of degree n in Tschirnhausen form can be identified with \mathbb{C}^{n-1} ; let the coordinates in \mathbb{C}^{n-1} be denoted by $\underline{a}_2, \underline{a}_3, \dots, \underline{a}_n$. The set

$$K := \bigcup_{k=2}^n \{(a_2, \dots, a_n) \in \mathbb{C}^{n-1} : a_k = 1, |a_j| \leq 1 \text{ for } j \neq k\}$$

is compact. For each point $p \in K$ there exists $\rho_p > 0$ such that $P_{\tilde{a}}$ splits on the open ball $B_{\rho_p}(p)$; cf. Section 4.2. Choose a finite subcover of K by open balls $B_{\rho_\delta}(p_\delta)$, $\delta \in \Delta$. Then there exists $\rho > 0$ so that for every $p \in K$ there is a $\delta \in \Delta$ such that $B_\rho(p) \subseteq B_{\rho_\delta}(p_\delta)$. Fix a universal positive constant B satisfying

$$(8.3) \quad B < \min \left\{ \frac{1}{3}, \frac{\rho}{3n^2 2^n} \right\}.$$

8.3. First splitting. Fix $t_0 \in (\alpha, \beta)$ and $k \in \{2, \dots, n\}$ such that (5.2) holds, i.e.,

$$(8.4) \quad |\tilde{a}_k(t_0)|^{1/k} = \max_{2 \leq j \leq n} |\tilde{a}_j(t_0)|^{1/j} \neq 0$$

This is possible unless $\tilde{a} \equiv 0$ in which case nothing is to prove. Choose a maximal open interval $I \subseteq (\alpha, \beta)$ containing t_0 such that we have (5.8), i.e.,

$$(8.5) \quad M|I| + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \leq B|\tilde{a}_k(t_0)|^{1/k},$$

with M given by (5.9).

In particular, all conclusions of Section 5 hold true.

Consider the point $p = \underline{a}(t_0)$, where \underline{a} is the curve defined in (5.7). By (8.4), $p \in K$ and thus there exists $\delta \in \Delta$ such that $B_\rho(p) \subseteq B_{\rho_\delta}(p_\delta)$. By Lemma 6 and by (8.3), the length of the curve $\underline{a}|_I$ is bounded by ρ . It follows that we have a splitting on I ,

$$P_{\tilde{a}}(t) = P_b(t)P_{b^*}(t), \quad t \in I.$$

By (4.2), the coefficients b_i of P_b are of the form

$$b_i = \tilde{a}_k^{i/k} \psi_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 1, \dots, \deg P_b,$$

and after Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde{b}}$, see (4.3),

$$\tilde{b}_i = \tilde{a}_k^{i/k} \tilde{\psi}_i(\tilde{a}_k^{-2/k} \tilde{a}_2, \dots, \tilde{a}_k^{-n/k} \tilde{a}_n), \quad i = 2, \dots, \deg P_b,$$

where ψ_i , respectively, $\tilde{\psi}_i$, are analytic functions all whose partial derivatives are separately bounded on $B_\rho(p)$. (Similar formulas hold for b_i^* and \tilde{b}_i^* .)

In summary, the restriction of the curve of polynomials $P_{\tilde{a}}$ to the interval I satisfies all assumptions and thus all conclusions of Sections 5 and 6.

It follows that the assumptions of the following proposition are satisfied.

Proposition 3. *Let $I \subseteq \mathbb{R}$ be a bounded open interval and let $P_{\tilde{a}}$ be a monic polynomial in Tschirnhausen form with coefficients of class $C^{\deg P_{\tilde{a}}-1,1}(\bar{I})$. Let $t_0 \in I$ and $k \in \{2, \dots, \deg P_{\tilde{a}}\}$ be such that*

$$(1) \quad |\tilde{a}_k(t_0)|^{1/k} = \max_{2 \leq j \leq \deg P_{\tilde{a}}} |\tilde{a}_j(t_0)|^{1/j} \neq 0,$$

- (2) $\sum_{j=2}^{\deg P_{\tilde{a}}} \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \leq B|\tilde{a}_k(t_0)|^{1/k}$ for some constant $B < 1/3$,
- (3) $|\tilde{a}_j^{(s)}(t)| \leq C|I|^{-s}|\tilde{a}_k(t_0)|^{j/k}$ for all $t \in I$, $j = 2, \dots, \deg P_{\tilde{a}}$, and $s = 1, \dots, \deg P_{\tilde{a}}$, and some constant $C = C(\deg P_{\tilde{a}})$.
- (4) Assume that $P_{\tilde{a}}$ splits on I , i.e., $P_{\tilde{a}}(t) = P_b(t)P_{b^*}(t)$ for $t \in I$.

Then every continuous root $\mu \in C^0(I)$ of $P_{\tilde{b}}$ is absolutely continuous and satisfies

$$(8.6) \quad \|\mu'\|_{L^p(I)} \leq C \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)} \right),$$

for all $1 \leq p < (\deg P_{\tilde{a}})'$ and a constant C depending only on $\deg P_{\tilde{a}}$ and p .

In this proposition and from now on we apply the following convention:

Any dependence of constants on parameters of the universal splitting, like ρ , $\tilde{\psi}_i$, etc., will no longer be explicitly stated. For simplicity it will henceforth be subsumed by saying that the constants depend on the degree of the polynomials. Universal constants will be denoted by C and may vary from line to line.

We shall prove this proposition by induction on the degree. The assumptions of the proposition amount exactly to the assumptions (5.1)–(5.3), (5.10), and (6.1)–(6.3). Thus we may rely on all conclusions of Sections 5 and 6.

8.4. Second splitting. By (5.5), \tilde{a}_k does not vanish on I , and thus b_i and \tilde{b}_i belong to $C^{n-1,1}(\bar{I})$. Let us set

$$I' := I \setminus \{t \in I : \tilde{b}_2(t) = \dots = \tilde{b}_{\deg P_b}(t) = 0\}.$$

For each $t_1 \in I'$ there is $\ell \in \{2, \dots, \deg P_b\}$ such that (6.7) holds, i.e.,

$$|\tilde{b}_\ell(t_1)|^{1/\ell} = \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(t_1)|^{1/i} \neq 0,$$

and, by Section 6.2, there is an open interval $J = J(t_1)$, $t_1 \in J \subseteq I'$, such that (6.9), i.e.,

$$|J||I|^{-1}|\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J)} = D|\tilde{b}_\ell(t_1)|^{1/\ell}.$$

The universal constant D can be chosen sufficiently small such that on J we have a splitting

$$P_b(t) = P_c(t)P_{c^*}(t), \quad t \in J;$$

in fact, it suffices to choose

$$(8.7) \quad D < \min \left\{ \frac{1}{3}, \frac{\sigma}{3(\deg P_b)^2 2^{\deg P_b}}, C_1^{-1} \right\},$$

where C_1 is the constant in (6.8) and where σ is the analogue of ρ in Section 8.2 for a cover of

$$\bigcup_{\ell=2}^{\deg P_b} \{(\underline{b}_2, \dots, \underline{b}_{\deg P_b}) \in \mathbb{C}^{\deg P_b-1} : \underline{b}_\ell = 1, |\underline{b}_i| \leq 1 \text{ for } i \neq \ell\}, \quad \underline{b}_i := \tilde{b}_\ell^{-i/\ell} \tilde{b}_i.$$

This follows from Lemma 9 and the arguments in Sections 8.2 and 8.3 applied to $P_{\tilde{b}}$.

By Proposition 2, we may conclude that there is a countable family $\{J_\gamma\}$ of open intervals $J_\gamma \subseteq I'$, of points $t_\gamma \in J_\gamma$, and of integers $\ell_\gamma \in \{2, \dots, \deg P_b\}$ satisfying

$$(8.8) \quad |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} = \max_{2 \leq i \leq \deg P_b} |\tilde{b}_i(t_\gamma)|^{1/i} \neq 0,$$

$$(8.9) \quad |J_\gamma| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J_\gamma)} = D |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma},$$

$$(8.10) \quad P_b(t) = P_{c_\gamma}(t) P_{c_\gamma}^*(t), \quad t \in J_\gamma,$$

$$(8.11) \quad \bigcup_{\gamma} J_\gamma = I', \quad \sum_{\gamma} |J_\gamma| \leq 2|I'|.$$

In particular, for every γ , the polynomial $P_b(t) = P_{c_\gamma}(t) P_{c_\gamma}^*(t)$, $t \in J_\gamma$, satisfies the assumptions of Proposition 3; indeed, (3) corresponds to (6.15).

8.5. Inductive step. Let $\mu \in C^0(I)$ be a continuous root of P_b . We may assume without loss of generality that in J_γ ,

$$(8.12) \quad \tilde{\mu}(t) := \mu(t) + \frac{c_{\gamma 1}(t)}{\deg P_{c_\gamma}}, \quad t \in J_\gamma,$$

is a root of $P_{\tilde{c}_\gamma}$. Since $\deg P_{\tilde{c}_\gamma} < \deg P_b < \deg P_a$, the induction hypothesis implies that $\tilde{\mu}$ is absolutely continuous and satisfies

$$(8.13) \quad \|\tilde{\mu}'\|_{L^p(J_\gamma)} \leq C \left(\| |J_\gamma|^{-1} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} \|_{L^p(J_\gamma)} + \sum_{h=2}^{\deg P_{c_\gamma}} \|(\tilde{c}_{\gamma h}^{1/h})'\|_{L^p(J_\gamma)} \right),$$

for all $1 \leq p < (\deg P_b)'$, for a constant C depending only on $\deg P_b$ and p .

8.6. L^p -estimates on I . By Section 4.2, the coefficients $c_{\gamma h}$ of P_{c_γ} are of the form

$$c_{\gamma h} = \tilde{b}_{\ell_\gamma}^{h/\ell_\gamma} \theta_h(\tilde{b}_{\ell_\gamma}^{-2/\ell_\gamma} \tilde{b}_2, \dots, \tilde{b}_{\ell_\gamma}^{-\deg P_b/\ell_\gamma} \tilde{b}_{\deg P_b}), \quad h = 1, \dots, \deg P_{c_\gamma},$$

and after Tschirnhausen transformation $P_{c_\gamma} \rightsquigarrow P_{\tilde{c}_\gamma}$, see (4.3),

$$\tilde{c}_{\gamma h} = \tilde{b}_{\ell_\gamma}^{h/\ell_\gamma} \tilde{\theta}_h(\tilde{b}_{\ell_\gamma}^{-2/\ell_\gamma} \tilde{b}_2, \dots, \tilde{b}_{\ell_\gamma}^{-\deg P_b/\ell_\gamma} \tilde{b}_{\deg P_b}), \quad h = 2, \dots, \deg P_{c_\gamma},$$

where θ_h , respectively, $\tilde{\theta}_h$, are analytic functions all whose partial derivatives are separately bounded. (Similar formulas hold for $c_{\gamma h}^*$ and $\tilde{c}_{\gamma h}^*$.) By (6.12), \tilde{b}_{ℓ_γ} does not vanish on J_γ and thus $c_{\gamma h}$ and $\tilde{c}_{\gamma h}$ belong to $C^{\deg P_a - 1, 1}(\overline{J_\gamma})$. Analogously to (6.4) we find that, for $t \in J_\gamma$, $h = 2, \dots, \deg P_{c_\gamma}$, and $s = 1, \dots, \deg P_a$,

$$|\tilde{c}_{\gamma h}^{(s)}(t)| \leq C |J_\gamma|^{-s} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{h/\ell_\gamma},$$

where $C = C(\deg P_a)$. Together with (3.3), it implies

$$\begin{aligned} \|(\tilde{c}_{\gamma h}^{1/h})'\|_{h', w, J_\gamma} &\leq C(h) \max \left\{ \left(\text{Lip}_{J_\gamma}(\tilde{c}_{\gamma h}^{(h-1)}) \right)^{1/h} |J_\gamma|^{1/h'}, \|\tilde{c}_{\gamma h}'\|_{L^\infty(J_\gamma)}^{1/h} \right\} \\ &\leq C |J_\gamma|^{-1+1/h'} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}, \end{aligned}$$

where the constant C depends only on $\deg P_{\tilde{a}}$. Thus,

$$\|(\tilde{c}_{\gamma h}^{1/h})'\|_{h',w,J_\gamma}^* \leq C|J_\gamma|^{-1}|\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma},$$

and so, in view of (2.3), for all p , $1 \leq p < (\deg P_{c_\gamma})'$,

$$\sum_{h=2}^{\deg P_{c_\gamma}} \|(\tilde{c}_{\gamma h}^{1/h})'\|_{L^p(J_\gamma)}^* \leq C|J_\gamma|^{-1}|\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma},$$

for a constant C that depends only on $\deg P_{\tilde{a}}$ and p . Consequently, by (8.9) and (2.2),

$$\begin{aligned} & \| |J_\gamma|^{-1} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} \|_{L^p(J_\gamma)}^* + \sum_{h=2}^{\deg P_{c_\gamma}} \|(\tilde{c}_{\gamma h}^{1/h})'\|_{L^p(J_\gamma)}^* \\ & \leq (1+C)|J_\gamma|^{-1} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} \\ & = (1+C)D^{-1} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^1(J_\gamma)}^* + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J_\gamma)}^* \right) \\ & \leq (1+C)D^{-1} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J_\gamma)}^* + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J_\gamma)}^* \right) \end{aligned}$$

and therefore

$$\begin{aligned} & \| |J_\gamma|^{-1} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} \|_{L^p(J_\gamma)}^p + \sum_{h=2}^{\deg P_{c_\gamma}} \|(\tilde{c}_{\gamma h}^{1/h})'\|_{L^p(J_\gamma)}^p \\ (8.14) \quad & \leq CD^{-p} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J_\gamma)}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J_\gamma)}^p \right), \end{aligned}$$

for a constant C that depends only on $\deg P_{\tilde{a}}$ and p .

Furthermore, the analogue of (6.5) gives

$$\|c'_{\gamma 1}\|_{L^\infty(J_\gamma)} \leq C|J_\gamma|^{-1} |\tilde{b}_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma},$$

where $C = C(\deg P_{\tilde{a}})$. Thus, by (8.9) and (2.2),

$$\|c'_{\gamma 1}\|_{L^p(J_\gamma)}^* \leq CD^{-1} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J_\gamma)}^* + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J_\gamma)}^* \right)$$

and hence

$$(8.15) \quad \|c'_{\gamma 1}\|_{L^p(J_\gamma)}^p \leq CD^{-p} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J_\gamma)}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J_\gamma)}^p \right),$$

for a constant C that depends only on $\deg P_{\tilde{a}}$ and p .

Hence, by (8.11), (8.13), and (8.14),

$$\begin{aligned}
 \sum_{\gamma} \|\tilde{\mu}'\|_{L^p(J_{\gamma})}^p &\leq CD^{-p} \sum_{\gamma} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(J_{\gamma})}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J_{\gamma})}^p \right) \\
 (8.16) \quad &\leq 2CD^{-p} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^p \right).
 \end{aligned}$$

Similarly, with (8.15) we get

$$(8.17) \quad \sum_{\gamma} \|c'_{\gamma 1}\|_{L^p(J_{\gamma})}^p \leq CD^{-p} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^p \right).$$

By (8.11), (8.12), (8.16), and (8.17), we may conclude that μ is absolutely continuous on I' and

$$\|\mu'\|_{L^p(I')}^p \leq \sum_{\gamma} \|\mu'\|_{L^p(J_{\gamma})}^p \leq CD^{-p} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)}^p + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^p \right),$$

and hence

$$\|\mu'\|_{L^p(I')} \leq CD^{-1} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)} \right),$$

for a constant C that depends only on $\deg P_{\tilde{a}}$ and p . Since μ vanishes on $I \setminus I'$, Lemma 1 implies that μ is absolutely continuous on I and

$$\|\mu'\|_{L^p(I)} \leq CD^{-1} \left(\| |I|^{-1} |\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)} \right).$$

This completes the proof of Proposition 3, since $C = C(\deg P_{\tilde{a}}, p)$ and $D = D(\deg P_{\tilde{a}})$ by (8.7).

8.7. End of proof of Theorem 1. We have seen in Section 8.3 that for a polynomial $P_{\tilde{a}}$ in Tschirnhausen form satisfying (8.4) and (8.5) the assumptions of Proposition 3 hold with the constant B fulfilling (8.3).

Let $\lambda \in C^0((\alpha, \beta))$ be a continuous root of $P_{\tilde{a}}$. We may assume without loss of generality that in I , it is a root of P_b . Then it has the form

$$(8.18) \quad \lambda(t) = -\frac{b_1(t)}{\deg P_b} + \mu(t), \quad t \in I,$$

where μ is a continuous root of $P_{\tilde{b}}$. By Proposition 3, μ is absolutely continuous on I and satisfies (8.6). Let us estimate the right-hand side of (8.6).

By Lemma 8, we have (6.4), and thus together with (3.3),

$$\begin{aligned}
 \|(\tilde{b}_i^{1/i})'\|_{i', w, I} &\leq C(i) \max \left\{ (\text{Lip}_I(\tilde{b}_i^{(i-1)}))^{1/i} |I|^{1/i'}, \|\tilde{b}_i'\|_{L^\infty(I)}^{1/i} \right\} \\
 &\leq C(n) |I|^{-1+1/i'} |\tilde{a}_k(t_0)|^{1/k}.
 \end{aligned}$$

Hence

$$\|(\tilde{b}_i^{1/i})'\|_{i',w,I}^* \leq C(n)|I|^{-1}|\tilde{a}_k(t_0)|^{1/k}.$$

Since $n' < \min_{2 \leq i \leq \deg P_b} i'$ and by (2.3), we get for all p , $1 \leq p < n'$,

$$\sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^* \leq C|I|^{-1}|\tilde{a}_k(t_0)|^{1/k},$$

where the constant C depends only on n and p . It follows that

$$(8.19) \quad \| |I|^{-1}|\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)}^* + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^* \leq (1+C)|I|^{-1}|\tilde{a}_k(t_0)|^{1/k}.$$

At this stage we distinguish the following two cases:

(i) Either we have equality in (8.5), i.e.,

$$(8.20) \quad M|I| + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} = B|\tilde{a}_k(t_0)|^{1/k}.$$

(ii) Or $I = (\alpha, \beta)$ and

$$(8.21) \quad M|I| + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} < B|\tilde{a}_k(t_0)|^{1/k}.$$

Case (i). In this cases we can estimate (8.19) by (8.20) and obtain

$$\begin{aligned} & \| |I|^{-1}|\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)}^* + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)}^* \\ & \leq CB^{-1} \left(M\|1\|_{L^1(I)}^* + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1(I)}^* \right) \quad \text{by (8.20)} \\ & \leq CB^{-1} \left(M\|1\|_{L^p(I)}^* + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)}^* \right) \quad \text{by (2.2)} \end{aligned}$$

and therefore

$$(8.22) \quad \begin{aligned} & \| |I|^{-1}|\tilde{a}_k(t_0)|^{1/k} \|_{L^p(I)} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(I)} \\ & \leq CB^{-1} \left(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)} \right), \end{aligned}$$

for a constant C that depends only on n and p .

Thus, by (8.6) and (8.22),

$$(8.23) \quad \|\mu'\|_{L^p(I)} \leq CB^{-1} \left(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)} \right).$$

Similarly, by (2.2), (6.5), (8.20), and (8.22),

$$(8.24) \quad \|b'_1\|_{L^p(I)} \leq CB^{-1} \left(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)} \right).$$

In view of (8.18), (8.23), and (8.24) we obtain that λ is absolutely continuous on I and

$$\|\lambda'\|_{L^p(I)} \leq CB^{-1} \left(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)} \right).$$

The constant M given (5.9) depends on t_0 ; thus we set

$$(8.25) \quad \tilde{A} := \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{C^{n-1,1}([\alpha, \beta])}^{1/j}$$

and estimate M by

$$\begin{aligned} M &= \max_{2 \leq j \leq n} (\text{Lip}_I(\tilde{a}_j^{(n-1)}))^{1/n} |\tilde{a}_k(t_0)|^{(n-j)/(kn)} \\ &\leq \max_{2 \leq j \leq n} \tilde{A}^{j/n} \tilde{A}^{(n-j)/n} = \tilde{A}. \end{aligned}$$

Thus,

$$(8.26) \quad \|\lambda'\|_{L^p(I)} \leq CB^{-1} \left(\tilde{A}\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p(I)} \right).$$

Case (ii). In this case we have a splitting $P_a = P_b P_{b^*}$ on the whole interval $I = (\alpha, \beta)$; cf. Section 8.3. Thus, (8.19) becomes

$$\begin{aligned} &\|(\beta - \alpha)^{-1} |\tilde{a}_k(t_0)|^{1/k}\|_{L^p((\alpha, \beta))} + \sum_{i=2}^{\deg P_b} \|(\tilde{b}_i^{1/i})'\|_{L^p((\alpha, \beta))} \\ &\leq C(\beta - \alpha)^{-1+1/p} |\tilde{a}_k(t_0)|^{1/k} \\ &\leq C(\beta - \alpha)^{-1+1/p} \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{L^\infty((\alpha, \beta))}^{1/j} \end{aligned}$$

Similarly, (6.5) implies

$$\|b'_1\|_{L^p((\alpha, \beta))} \leq C(\beta - \alpha)^{-1+1/p} \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{L^\infty((\alpha, \beta))}^{1/j}.$$

In view of (8.18) and (8.6) we obtain that λ is absolutely continuous on (α, β) and

$$(8.27) \quad \|\lambda'\|_{L^p((\alpha, \beta))} \leq C(\beta - \alpha)^{-1+1/p} \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{L^\infty((\alpha, \beta))}^{1/j},$$

where $C = C(n, p)$.

Gluing the estimates. In Case (ii) the bound (8.27) holds on the whole interval; no gluing is required. Hence, if there is at least one point t_0 in (α, β) at which Case (ii) occurs, we are done.

Let us assume that at all points in (α, β) , that do not satisfy (8.2), Case (i) occurs. In analogy to Section 8.4, we can cover the complement in (α, β) of the points t satisfying (8.2) by a countable family \mathcal{I} of open intervals I on which (8.26) holds and such that $\sum_{I \in \mathcal{I}} |I| \leq 2(\beta - \alpha)$. Since λ vanishes on the points t satisfying (8.2), we can apply Lemma 1 and obtain that λ is absolutely continuous on (α, β) and satisfies

$$\|\lambda'\|_{L^p((\alpha, \beta))} \leq CB^{-1} \left(\tilde{A} \|1\|_{L^p((\alpha, \beta))} + \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^p((\alpha, \beta))} \right).$$

By (3.3), we may conclude that λ is absolutely continuous on (α, β) and satisfies (8.28)

$$\|\lambda'\|_{L^p((\alpha, \beta))} \leq CB^{-1} \left(\tilde{A}(\beta - \alpha)^{1/p} + \sum_{j=2}^n \max \left\{ (\text{Lip}_{(\alpha, \beta)}(\tilde{a}_j^{(j-1)}))^{1/j} (\beta - \alpha)^{1-1/j}, \|\tilde{a}_j'\|_{L^\infty((\alpha, \beta))}^{1/j} \right\} \right),$$

where $C = C(n, p)$ and $B = B(n)$ by (8.3).

Remarks. (1) We can avoid the distinction of cases in Section 8.7 if we require that the constant B also satisfies

$$(8.29) \quad B \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{L^\infty((\alpha, \beta))}^{1/j} \leq M(\beta - \alpha).$$

which enforces Case (i). Then, however, the factor B^{-1} that appears in (8.28) blows up as $\beta - \alpha \rightarrow 0$.

(2) Also the bound in Case (ii) for $\|\lambda'\|_{L^p((\alpha, \beta))}$ in (8.27) tends to infinity if $\beta - \alpha \rightarrow 0$ unless $p = 1$.

(3) A sufficient condition for the elimination of this blow-up phenomenon is the following. Assume that for all $j = 2, \dots, n$ there is a point $s = s(j) \in (\alpha, \beta)$ such that $\tilde{a}_j(s) = 0$. In that case we have, for $t \in (\alpha, \beta)$,

$$|\tilde{a}_j^{1/j}(t)| = \left| \int_s^t (\tilde{a}_j^{1/j})' d\tau \right| \leq \|(\tilde{a}_j^{1/j})'\|_{L^1((\alpha, \beta))}$$

and hence

$$(8.30) \quad \max_{2 \leq j \leq n} \|\tilde{a}_j\|_{L^\infty((\alpha, \beta))}^{1/j} \leq \sum_{j=2}^n \|(\tilde{a}_j^{1/j})'\|_{L^1((\alpha, \beta))}.$$

Since $B < 1$ (by (8.3)), (8.30) enforces equality in (8.5) and thus only Case (i) occurs. Since the constant B is only restricted by (8.3) it is universal.

8.8. The uniform bound (1.2). The bounds (8.28) and (8.27) imply

$$(8.31) \quad \|\lambda'\|_{L^p((\alpha, \beta))} \leq C(n, p) \max\{1, (\beta - \alpha)^{1/p}, (\beta - \alpha)^{-1+1/p}\} \tilde{A},$$

where \tilde{A} is given by (8.25).

In order to get the bound in terms of the a_j (i.e., *before* Tschirnhausen transformation) let $\hat{\lambda} := \lambda - a_1/n$ and set

$$A := \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}([\alpha, \beta])}^{1/j}.$$

Then

$$\|\hat{\lambda}'\|_{L^p((\alpha, \beta))} \leq \|\lambda'\|_{L^p((\alpha, \beta))} + (1/n)\|a_1'\|_{L^p((\alpha, \beta))}$$

and

$$\|a_1'\|_{L^p((\alpha, \beta))} \leq (\beta - \alpha)^{1/p} \|a_1'\|_{L^\infty((\alpha, \beta))}.$$

Observe that

$$\tilde{A} \leq C(n)A,$$

by the weighted homogeneity of the formulas (4.1). Hence, by (8.31),

$$\|\hat{\lambda}'\|_{L^p((\alpha, \beta))} \leq C(n, p) \max\{1, (\beta - \alpha)^{1/p}, (\beta - \alpha)^{-1+1/p}\}A,$$

that is (1.2). The proof of Theorem 1 is complete.

9. PROOF OF THEOREM 2

Theorem 2 follows from Theorem 1 by the arguments given in the proof of [29, Theorem 4.1]. We provide full details in order to see that the constant in the bound (1.3) does not depend on V ; this will be important in forthcoming work.

Proof of Theorem 2. By Theorem 1, λ is absolutely continuous along affine lines parallel to the coordinate axes (restricted to V). So λ possesses the partial derivatives $\partial_i \lambda$, $i = 1, \dots, m$, which are defined almost everywhere and are measurable.

Set $x = (t, y)$, where $t = x_1$, $y = (x_2, \dots, x_m)$, and let V_1 be the orthogonal projection of V on the hyperplane $\{x_1 = 0\}$. For each $y \in V_1$ we denote by $V^y := \{t \in \mathbb{R} : (t, y) \in V\}$ the corresponding section of V ; note that V^y is open in \mathbb{R} .

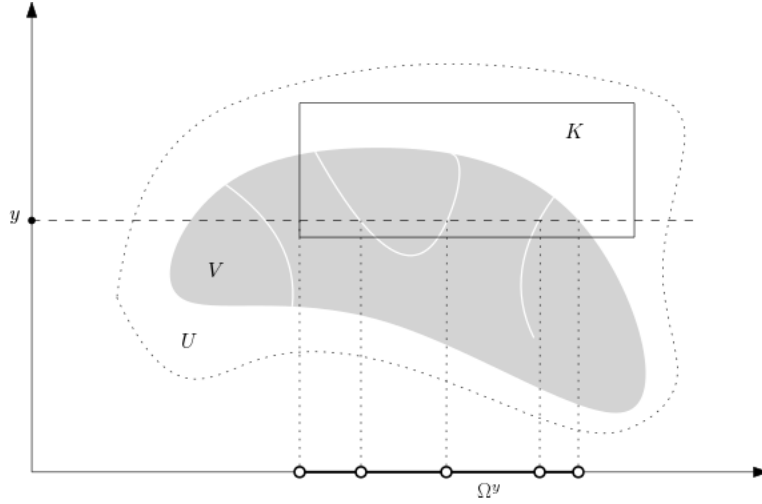
We may cover \bar{V} by finitely many open boxes $K = I_1 \times \dots \times I_m$ contained in U . Let K be fixed and set $L = I_2 \times \dots \times I_m$. Fix $y \in V_1 \cap L$ and let λ_j^y , $j = 1, \dots, n$, be a continuous system of the roots of $P_a(\cdot, y)$ on $\Omega^y := V^y \cap I_1$ such that $\lambda(\cdot, y) = \lambda_1^y$; it exists since $\lambda(\cdot, y)$ can be completed to a continuous system of the roots of $P_a(\cdot, y)$ on each connected component of Ω^y by [37, Lemma 6.17]. Our goal is to bound

$$\|\partial_t \lambda(\cdot, y)\|_{L^p(\Omega^y)} = \|(\lambda_1^y)'\|_{L^p(\Omega^y)}$$

uniformly in $y \in V_1 \cap L$.

To this end let \mathcal{C}^y denote the set of connected components J of the open subset $\Omega^y \subseteq \mathbb{R}$. For each $J \in \mathcal{C}^y$ we extend the system of roots $\lambda_j^y|_J$, $j = 1, \dots, n$, continuously to I_1 , i.e., we choose continuous functions $\lambda_j^{y,J}$, $j = 1, \dots, n$, on I_1 such that $\lambda_j^{y,J}|_J = \lambda_j^y|_J$ for all j and

$$P_a(t, y)(Z) = \prod_{j=1}^n (Z - \lambda_j^{y,J}(t)), \quad t \in I_1.$$



This is possible since $\lambda_j^y|_J$ has a continuous extension to the endpoints of the (bounded) interval J , by [21, Lemma 4.3], and can then be extended on the left and on the right of J by a continuous system of the roots of $P_a(\cdot, y)$ after suitable permutations.

By Theorem 1, for each $y \in V_1 \cap L$, $J \in \mathcal{C}^y$, and $j = 1, \dots, n$, the function $\lambda_j^{y,J}$ is absolutely continuous on I_1 and $(\lambda_j^{y,J})' \in L^p(I_1)$, for $1 \leq p < n/(n-1)$, with

$$(9.1) \quad \|(\lambda_j^{y,J})'\|_{L^p(I_1)} \leq C(n, p, |I_1|) \max_{1 \leq i \leq n} \|a_i\|_{C^{n-1,1}(\overline{K})}^{1/i}.$$

Let $J, J_0 \in \mathcal{C}^y$ be arbitrary. By [29, Lemma 3.6], $(\lambda_j^y)'$ as well as $(\lambda_j^{y,J_0})'$ belong to $L^p(J)$ and we have

$$\sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(J)}^p = \sum_{j=1}^n \|(\lambda_j^{y,J})'\|_{L^p(J)}^p = \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|_{L^p(J)}^p.$$

Thus,

$$\begin{aligned} \sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(\Omega^y)}^p &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(J)}^p \\ &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|_{L^p(J)}^p \\ &= \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|_{L^p(\Omega^y)}^p \\ &\leq \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|_{L^p(I_1)}^p. \end{aligned}$$

In particular, by (9.1),

$$\|\partial_t \lambda(\cdot, y)\|_{L^p(\Omega^y)} = \|(\lambda_1^y)'\|_{L^p(\Omega^y)} \leq C(n, p, K) \max_{1 \leq i \leq n} \|a_i\|_{C^{n-1,1}(\overline{K})}^{1/i},$$

and so, by Fubini's theorem,

$$\begin{aligned} \int_{V \cap K} |\partial_1 \lambda(x)|^p dx &= \int_{V_1 \cap L} \int_{\Omega^y} |\partial_1 \lambda(t, y)|^p dt dy \\ &\leq \left(C(n, p, K) \max_{1 \leq i \leq n} \|a_i\|_{C^{n-1,1}(\overline{K})}^{1/i} \right)^p \int_{V_1 \cap L} dy, \end{aligned}$$

and thus

$$\|\partial_1 \lambda\|_{L^p(V \cap K)} \leq C(n, p, K) \max_{1 \leq i \leq n} \|a_i\|_{C^{n-1,1}(\overline{K})}^{1/i}.$$

The other partial derivatives $\partial_i \lambda$, $i \geq 2$, are treated analogously. This implies (1.3), where W is the (finite) union of the boxes K . \square

10. APPLICATIONS

In this section we present two applications of our main results, Theorems 1 and 2. First we improve upon a result due to Spagnolo [40] on local solvability of certain systems of pseudodifferential equations. Secondly, we obtain a lifting theorem for differentiable mappings into orbit spaces of finite group representations.

10.1. Local solvability of pseudodifferential equations. In [40] Spagnolo proved that the pseudodifferential $n \times n$ system

$$(10.1) \quad u_t + iA(t, D_x)u + B(t, D_x)u = f(t, x), \quad (t, x) \in I \times U \subseteq \mathbb{R} \times \mathbb{R}^m,$$

where $A \in C^\infty(I, S^1(\mathbb{R}^m))^{n \times n}$, $B \in C^0(I, S^0(\mathbb{R}^m))^{n \times n}$ are matrix symbols of order 1 and 0, respectively, and $A(t, \xi)$ is homogeneous of degree 1 in ξ for $|\xi| \geq 1$, is locally solvable in the Gevrey class G^s for $1 \leq s \leq n/(n-1)$ and semi-globally solvable in G^s for $1 < s < n/(n-1)$ under the following assumptions: the eigenvalues of $A(t, \xi)$ admit a parameterization $\tau_1(t, \xi), \dots, \tau_n(t, \xi)$ such that each $\tau_j(t, \xi)$ is absolutely continuous in t , uniformly with respect to ξ , i.e.,

$$(\mathcal{A}_1) \quad |\partial_t \tau_j(t, \xi)| \leq \mu(t, \xi)(1 + |\xi|^2)^{1/2}, \quad \text{with } \mu(\cdot, \xi) \text{ equi-integrable on } I,$$

and for each ξ the imaginary parts of the $\tau_j(t, \xi)$ do not change sign for varying t and j , i.e.,

$$(\mathcal{A}_2) \quad \forall \xi \quad \text{either } \operatorname{Im} \tau_j(t, \xi) \geq 0, \quad \forall t, j, \quad \text{or } \operatorname{Im} \tau_j(t, \xi) \leq 0, \quad \forall t, j.$$

Theorem 1 implies that the assumption (\mathcal{A}_1) is always satisfied. Indeed, this follows by applying Theorem 1 to the characteristic polynomial of the matrix $(1 + |\xi|^2)^{-1/2} A(t, \xi)$ and noting that the entries of $(1 + |\xi|^2)^{-1/2} A(t, \xi)$ and its iterated partial derivatives with respect to t are globally bounded in ξ , since $A(t, \xi)$ is a symbol of order 1.

In particular, the scalar equation

$$(10.2) \quad \partial_t^n u + \sum_{j=1}^n a_j(t, D_x) \partial_t^{n-j} u = f(t, x),$$

where u, f are scalar functions and $a_j(t, D_x)$ is a pseudodifferential operator of order j with principal symbol $a_j^0(t, \xi)$ smooth in t , is locally solvable in G^s for $1 \leq s \leq n/(n-1)$ and

semi-globally solvable in G^s for $1 < s < n/(n-1)$ provided that the roots $\tau_1(t, \xi), \dots, \tau_n(t, \xi)$ of

$$(iZ)^n + \sum_{j=1}^n a_j^0(t, \xi)(iZ)^{n-j} = 0$$

satisfy assumption (\mathcal{A}_2) ; cf. [40, Corollary 2].

A crucial tool in the proof is the technique of quasi-diagonalization for a Sylvester matrix, introduced by [18] for weakly hyperbolic problems and then refined by [13].

Actually, by Theorem 1, the above conclusions hold provided that the matrix symbol $A(t, \xi)$ is just of class $C^{n-1,1}$ in time t .

Theorem 4. *The pseudodifferential $n \times n$ system (10.1), where $A \in C^{n-1,1}(I, S^1(\mathbb{R}^m))^{n \times n}$, $B \in C^0(I, S^0(\mathbb{R}^m))^{n \times n}$, and $A(t, \xi)$ is homogeneous of degree 1 in ξ for $|\xi| \geq 1$, is locally solvable in the Gevrey class G^s for $1 \leq s \leq n/(n-1)$ and semi-globally solvable in G^s for $1 < s < n/(n-1)$ provided that the eigenvalues $\tau_1(t, \xi), \dots, \tau_n(t, \xi)$ of $A(t, \xi)$ satisfy (\mathcal{A}_2) .*

Proof. Theorem 1 implies (\mathcal{A}_1) provided that $A(t, \xi)$ is $C^{n-1,1}$ in t . Then the proof in [40] yields the result. \square

10.2. Lifting mappings from orbit spaces. Let G be a finite group and let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G on a complex finite dimensional complex vector space V . By Hilbert's theorem, the algebra $\mathbb{C}[V]^G$ of G -invariant polynomials on V is finitely generated. We consider the categorical quotient $V//G$, i.e., the affine algebraic variety with coordinate ring $\mathbb{C}[V]^G$, and the morphism $\pi : V \rightarrow V//G$ defined by the embedding $\mathbb{C}[V]^G \rightarrow \mathbb{C}[V]$. Since G is finite, $V//G$ coincides with the orbit space V/G . Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{C}[V]^G$ with positive degrees d_1, \dots, d_n . Then we can identify π with the mapping of invariants $\sigma = (\sigma_1, \dots, \sigma_n) : V \rightarrow \sigma(V) \subseteq \mathbb{C}^n$ and the orbit space V/G with the image $\sigma(V)$.

Let $U \subseteq \mathbb{R}^m$ be open, $k \in \mathbb{N}$. Consider a mapping $f \in C^{k-1,1}(U, \sigma(V))$, i.e., f is of Hölder class $C^{k-1,1}$ as mapping $U \rightarrow \mathbb{C}^n$ with the image $f(U)$ contained in $\sigma(V) \subseteq \mathbb{C}^n$. We say that a mapping $\bar{f} : U \rightarrow V$ is a *lift of f over σ* if $f = \sigma \circ \bar{f}$. It is natural to ask how *regular* a lift of f can be chosen. This question is independent of the choice of generators of $\mathbb{C}[V]^G$, since any two choices differ by a polynomial diffeomorphism. This and similar problems were studied in [2], [21], [22], [23], [24], [26], [36], [30].

$$\begin{array}{ccccc}
 & & V & \xrightarrow{\quad} & G \\
 & \nearrow \bar{f} & \downarrow \sigma & & \\
 U & \xrightarrow{\quad f \quad} & \sigma(V) & \hookrightarrow & \mathbb{C}^n \\
 & & \parallel & & \\
 & & V/G & &
 \end{array}$$

The subject of this paper, i.e., optimal regularity of roots of polynomials, is just a special case of this problem: let the symmetric group S_n act on \mathbb{C}^n by permuting the coordinates. Then $\mathbb{C}[\mathbb{C}^n]^{S_n}$ is generated by the elementary symmetric polynomials $\sigma_j(z) =$

$\sum_{i_1 < \dots < i_j} z_{i_1} \cdots z_{i_j}$, $\mathbb{C}^n/S_n = \sigma(\mathbb{C}^n) = \mathbb{C}^n$, and $f : U \rightarrow \sigma(\mathbb{C}^n)$ amounts to a family of complex monic polynomials P_f with coefficients $(-1)^j f_j$, $j = 1, \dots, n$, in view of Vieta's formulas. Lifting f over σ precisely means choosing the roots of P_f .

As an application of our main Theorems 1 and 2 we obtain the following lifting result for finite groups. Following Noether's proof of Hilbert's theorem we associated a suitable polynomial and use the regularity result for its roots.

Theorem 5. *Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a complex finite dimensional representation of a finite group G . Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{C}[V]^G$. Decompose $V = \bigoplus_{i=1}^{\ell} V_i$ into irreducible subrepresentations of G , and let*

$$k := \max_{i=1, \dots, \ell} \min_{v \in V_i \setminus \{0\}} |Gv|.$$

Then:

- (1) *If $c \in C^{k-1,1}(I, \sigma(V))$, where $I \subseteq \mathbb{R}$ is a compact interval, then any continuous lift $\bar{c} \in C^0(I, V)$ of c is absolutely continuous and belongs to the Sobolev space $W^{1,p}(I, V)$ for every $1 \leq p < k/(k-1)$. If \mathcal{C} is a bounded subset of $C^{k-1,1}(I, \sigma(V))$, then $\bar{\mathcal{C}} := \{\bar{c} \in C^0(I, V) : \sigma \circ \bar{c} \in \mathcal{C}\}$ is bounded in $W^{1,p}(I, V)$ for every $1 \leq p < k/(k-1)$.*
- (2) *If $f \in C^{k-1,1}(U, \sigma(V))$, where $U \subseteq \mathbb{R}^m$ is open, and $\bar{f} \in C^0(\Omega, V)$ is a continuous lift of f on a relatively compact open subset $\Omega \Subset U$, then \bar{f} belongs to the Sobolev space $W^{1,p}(\Omega, V)$ for every $1 \leq p < k/(k-1)$. If \mathcal{F} is a bounded subset of $C^{k-1,1}(U, \sigma(V))$, then $\bar{\mathcal{F}} := \{\bar{f} \in C^0(\Omega, V) : \sigma \circ \bar{f} \in \mathcal{F}\}$ is bounded in $W^{1,p}(\Omega, V)$ for every $1 \leq p < k/(k-1)$.*

Note that there always exists a continuous lift \bar{c} of $c \in C^0(I, \sigma(V))$; see [26, Theorem 5.1].

Proof. By treating the irreducible subrepresentations separately, we may assume without loss of generality that ρ is irreducible. Fix a non-zero vector $v \in V$ such that $|Gv|$ is minimal. Choose a G -invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on V , and associate to $g \in G$ the linear form $\ell_g : V \rightarrow \mathbb{C}$ defined by $\ell_g(x) := \langle x, gv \rangle$. Choose a numbering of the left coset $G/G_v = \{g_1, \dots, g_k\}$, where $G_v = \{g \in G : gv = v\}$ and $k = |Gv|$, and set $\ell_i := \ell_{g_i}$ for $i = 1, \dots, k$. Then the action of G on G/G_v by left multiplication induces a permutation of the set $\{g_1, \dots, g_k\}$, and thus

$$a_j := (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \ell_{i_1} \cdots \ell_{i_j}, \quad j = 1, \dots, k,$$

are G -invariant polynomials on V . So $a_j = p_j \circ \sigma$ for polynomials $p_j \in \mathbb{C}[\mathbb{C}^n]$, and the polynomial $P_a \in \mathbb{C}[V]^G[Z]$ given by

$$P_{a(x)}(Z) = Z^k + \sum_{j=1}^k a_j(x) Z^{k-j} = \prod_{j=1}^k (Z - \ell_j(x)), \quad x \in V,$$

factors through the polynomial $P_p \in \mathbb{C}[\mathbb{C}^n][Z]$, i.e., $P_a = P_{p \circ \sigma}$. Applying Theorem 1 to $P_{p(c(t))}$, $t \in I$, we find that $t \mapsto \ell_i(\bar{c}(t)) = \langle \bar{c}(t), g_i v \rangle$, $i = 1, \dots, k$, belongs to $W^{1,p}(I)$ for each $1 \leq p < k/(k-1)$. Since ρ is irreducible, the orbit Gv spans V and (1) follows. Analogously, (2) follows from Theorem 2. \square

As a consequence one obtains a similar result for polar representations of reductive algebraic groups, since the lifting problem can be reduced to the action of the corresponding generalized Weyl group which is finite; cf. [26] or [36].

10.3. Multi-valued Sobolev functions. In [3] Almgren developed a theory of n -valued Sobolev functions and proved the existence of n -valued minimizers of the Dirichlet energy functional. See also [14] for simpler proofs.

An n -valued function is a function with values in the set of unordered n -tuples of points in \mathbb{R}^ℓ . The latter may be represented by the set

$$\mathcal{A}_n(\mathbb{R}^\ell) := \left\{ \sum_{i=1}^n \delta_{x_i} : x_i \in \mathbb{R}^\ell, i = 1, \dots, n \right\},$$

where δ_x denotes the Dirac δ -measure $\delta_x(E) = \chi_E(x)$ in \mathbb{R}^ℓ . It forms a complete metric space when endowed with the metric

$$d\left(\sum_{i=1}^n \delta_{x_i}, \sum_{i=1}^n \delta_{y_i}\right) := \min_{\sigma \in S_n} \left(\sum_{i=1}^n |x_i - y_{\sigma(i)}|^2 \right)^{1/2},$$

where S_n is the group of permutation of $\{1, \dots, n\}$. Almgren proved that there is an integer $N = N(n, \ell)$ and an injective mapping $\Delta : \mathcal{A}_n(\mathbb{R}^\ell) \rightarrow \mathbb{R}^N$ such that $\text{Lip}(\Delta) \leq 1$ and $\text{Lip}(\Delta^{-1}|_{\Delta(\mathcal{A}_n(\mathbb{R}^\ell))}) \leq C(n, \ell)$; moreover, there is a Lipschitz retraction of \mathbb{R}^N onto $\Delta(\mathcal{A}_n(\mathbb{R}^\ell))$.

One can use this biLipschitz embedding to define n -valued Sobolev functions: for open $U \subseteq \mathbb{R}^m$ and $1 \leq p \leq \infty$ define

$$W^{1,p}(U, \mathcal{A}_n(\mathbb{R}^\ell)) := \{f : U \rightarrow \mathcal{A}_n(\mathbb{R}^\ell) : \Delta \circ f \in W^{1,p}(U, \mathbb{R}^N)\}.$$

For an intrinsic definition see [14, Definition 0.5 and Theorem 2.4].

Let us identify $\mathbb{R}^2 \cong \mathbb{C}$. The elementary symmetric polynomials induce a bijective mapping $a : \mathcal{A}_n(\mathbb{C}) \rightarrow \mathbb{C}^n$, where \mathbb{C}^n is the space of monic complex polynomials P_a of degree n ,

$$a_j \left(\sum_{i=1}^n \delta_{z_i} \right) := (-1)^j \sum_{i_1 < \dots < i_j} z_{i_1} \cdots z_{i_j}.$$

In other words, monic complex polynomials of degree n are in one-to-one correspondence with its unordered n -tuples of roots.

Our Theorems 1 and 2 provide a sufficient condition for an n -valued function $U \rightarrow \mathcal{A}_n(\mathbb{C})$ to belong to the Sobolev space $W^{1,p}(U, \mathcal{A}_n(\mathbb{C}))$.

We shall use the following terminology. By a *parameterization* of an n -valued function $f : U \rightarrow \mathcal{A}_n(\mathbb{C})$ we mean a function $\varphi : U \rightarrow \mathbb{C}^n$ such that $f(x) = \sum_{i=1}^n \delta_{\varphi_i(x)}$ for all $x \in U$. If we denote by π the mapping $\pi : \mathbb{C}^n \rightarrow \mathcal{A}_n(\mathbb{C})$ which sends an ordered n -tuple (z_1, \dots, z_n) to the corresponding unordered n -tuple $\sum_{i=1}^n \delta_{z_i}$, then a parameterization of f amounts to a lift φ of f over π , i.e., $f = \pi \circ \varphi$. Note that π is a Lipschitz mapping with $\text{Lip}(\pi) = 1$.

Theorem 6. *Let $U \subseteq \mathbb{R}^m$ be open and let $f : U \rightarrow \mathcal{A}_n(\mathbb{C})$ be continuous. If $a \circ f \in C^{n-1,1}(U, \mathbb{C}^n)$, then $f \in W^{1,p}(V, \mathcal{A}_n(\mathbb{C}))$ for each relatively compact $V \Subset U$ and each $1 \leq p < n/(n-1)$, and*

$$\|\nabla(\Delta \circ f)\|_{L^p(V)} \leq C(m, n, p, \mathcal{K}, \Delta) \left(1 + \max_{1 \leq j \leq n} \|a_j \circ f\|_{C^{n-1,1}(\overline{V})}^{1/j}\right),$$

where \mathcal{K} is any finite cover of \overline{V} by open boxes $\prod_{i=1}^m (\alpha_i, \beta_i)$ contained in U and $W = \bigcup \mathcal{K}$.

Proof. We must show that $\Delta \circ f$ is an element of $W^{1,p}(V, \mathbb{R}^N)$. Clearly, $\Delta \circ f : U \rightarrow \mathbb{R}^N$ is continuous. The set V can be covered by finitely many open boxes $I = \prod_{i=1}^m (\alpha_i, \beta_i)$.

Let e_i be the i th standard unit vector in \mathbb{R}^m . Denote by I_i the orthogonal projection of I onto the hyperplane e_i^\perp . For each $y \in I_i$ we have $(\alpha_i, \beta_i) = \{t \in \mathbb{R} : y + te_i \in I\}$.

By Theorem 1, $(\alpha_i, \beta_i) \ni t \mapsto f(y + te_i)$ admits an absolutely continuous parameterization $\varphi_{i,y}$ such that

$$\|\varphi'_{i,y}\|_{L^p((\alpha_i, \beta_i))} \leq C(n, p, \beta_i - \alpha_i) \max_{1 \leq j \leq n} \|a_j \circ f\|_{C^{n-1,1}(\overline{I})}^{1/j}.$$

Thus, $(\alpha_i, \beta_i) \ni t \mapsto \Delta(f(y + te_i)) = \Delta(\pi(\varphi_{i,y}(t)))$ is absolutely continuous and

$$\|(\Delta \circ \pi \circ \varphi_{i,y})'\|_{L^p((\alpha_i, \beta_i))} \leq C(n, p, \beta_i - \alpha_i, \Delta) \left(1 + \max_{1 \leq j \leq n} \|a_j \circ f\|_{C^{n-1,1}(\overline{I})}^{1/j}\right),$$

since $\Delta \circ \pi$ is a Lipschitz mapping; cf. [28]. By Fubini's theorem,

$$\int_I |\partial_i(\Delta \circ f)|^p dx = \int_{I_i} \int_{\alpha_i}^{\beta_i} |(\Delta \circ \pi \circ \varphi_{i,y})'|^p dt dy,$$

and the statement follows. \square

$$\begin{array}{ccccc} & & & \mathbb{C}^n & \\ & \nearrow \varphi_{i,y} & & \downarrow \pi & \searrow \\ (\alpha_i, \beta_i) \hookrightarrow I & \xrightarrow{f} & \mathcal{A}_n(\mathbb{C}) & \hookrightarrow & \mathbb{R}^N \\ & \searrow & \uparrow a & \downarrow a^{-1} & \\ & & \mathbb{C}^n & & \end{array}$$

In particular, the roots of a polynomial P_a of degree n with coefficients $a_j \in C^{n-1,1}(U)$, $j = 1, \dots, n$, form an n -valued function $\lambda \in W_{\text{loc}}^{1,p}(U, \mathcal{A}_n(\mathbb{C}))$ for each $1 \leq p < n/(n-1)$; in fact, it is well-known that $\lambda : U \rightarrow \mathcal{A}_n(\mathbb{C})$ is continuous (cf. [19] or [32, Theorem 1.3.1]).

Theorem 6 implies that the push-forward

$$(a^{-1})_* : C^{n-1,1}(U, \mathbb{C}^n) \rightarrow \bigcap_{1 \leq p < n/(n-1)} W_{\text{loc}}^{1,p}(U, \mathcal{A}_n(\mathbb{C})).$$

is a bounded mapping.

We remark that much more is true in the case of *real* n -valued functions. In this situation the elementary symmetric polynomials induce a bijective mapping $a : \mathcal{A}_n(\mathbb{R}) \rightarrow H_n$, where H_n is a closed semialgebraic subset of \mathbb{R}^n , namely, the space of so-called *hyperbolic* polynomials of degree n . Then the mapping

$$(a^{-1})_* : C^{n-1,1}(U, H_n) \rightarrow C^{0,1}(U, \mathcal{A}_n(\mathbb{R})),$$

is bounded. It is easy to see that a continuous function $f : U \rightarrow \mathcal{A}_n(\mathbb{R})$ admits a continuous parameterization, for instance, by ordering the components increasingly which defines a

section θ of the projection $\pi : \mathbb{R}^n \rightarrow \mathcal{A}_n(\mathbb{R})$. Then we have a bounded mapping

$$(\theta \circ a^{-1})_* : C^{n-1,1}(U, H_n) \rightarrow C^{0,1}(U, \mathbb{R}^n).$$

All this essentially follows from Bronshtein's theorem [8]; see [31].

Remark 3. Let $\Phi : \mathcal{A}_n(\mathbb{C}) \rightarrow \mathcal{A}_n(\mathbb{R})$ be a Lipschitz function. If $f \in W^{1,p}(U, \mathcal{A}_n(\mathbb{C}))$, then $\Phi \circ f \in W^{1,p}(U, \mathcal{A}_n(\mathbb{R}))$ and it admits a parameterization $\theta \circ \Phi \circ f \in W^{1,p}(U, \mathbb{R}^n)$. This follows from the following diagram in which all vertical arrows are Lipschitz; the arrows in the lower row by Almgren's results, and θ is Lipschitz, since $d(x, y) = |\theta(x) - \theta(y)|$ for $z, w \in \mathcal{A}_n(\mathbb{R})$.

$$\begin{array}{ccccc}
 & & & & \mathbb{R}^n \\
 & & & \nearrow & \uparrow \pi \\
 U & \xrightarrow{f} & \mathcal{A}_n(\mathbb{C}) & \xrightarrow{\Phi} & \mathcal{A}_n(\mathbb{R}) \\
 & \searrow \Delta_2 & \uparrow & & \uparrow \theta \\
 & & \mathbb{R}^{N_2} & \xrightarrow{\quad} & \mathbb{R}^{N_1}
 \end{array}$$

Every Lipschitz function $\phi : \mathbb{C} \rightarrow \mathbb{R}$ induces a Lipschitz functions $\Phi : \mathcal{A}_n(\mathbb{C}) \rightarrow \mathcal{A}_n(\mathbb{R})$ by setting $\Phi(\sum_{i=1}^n \delta_{z_i}) := \sum_{i=1}^n \delta_{\phi(z_i)}$. In particular, by Theorem 6, the absolute values and real and imaginary parts of the roots of a monic polynomial P_a of degree n with coefficients in $C^{n-1,1}(U)$ admit continuous parameterization in $W_{\text{loc}}^{1,p}(U, \mathbb{R}^n)$ for each $1 \leq p < n/(n-1)$; but not simultaneously!

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ADAM PARUSIŃSKI: UNIV. NICE SOPHIA ANTIPOLIS, CNRS, LJAD, UMR 7351, 06108 NICE, FRANCE

E-mail address: `adam.parusinski@unice.fr`

ARMIN RAINER: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, A-1090 WIEN, AUSTRIA

E-mail address: `armin.rainer@univie.ac.at`